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Quantized Singularities in the Gravitational Field

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Abstract

The Cabibbo–Ferrari derivation of the Dirac charge quantization condition in electromagnetism is extended to the gravitational field. This is accomplished by use of the path-dependent formalism pioneered by Mandelstam. As a result, we find that the Bianchi identity generalizes to include a quantized, singular source term for the dual Riemann tensor. Under reasonable assumptions, this source term is proportional to the divergence of the energy-momentum tensor, leading to a quantized violation of local energy conservation. Specifically, it is found that the magnitude of the time rate of appearance of three-momentum in any volume of three-space must be an integer multiple of $3c^4/2G$. Some physical aspects of this energy nonconservation are briefly considered.

1. *Introduction*

In 1931 Dirac demonstrated that the existence of magnetic monopoles is consistent with quantum mechanics only if a quantization condition is imposed on the sources of the electromagnetic field; that is, electric and magnetic charge must be quantized. Since Dirac's pioneering work, this theme has been taken up by several authors, with notable contributions in particular from Schwinger (1966, 1968) and Cabibbo & Ferrari (1962). Using a variety of techniques, all of these authors arrive at essentially the same conclusions as Dirac.

On the other hand, examples of the application of analogous arguments to the gravitational field have been scant, presumably owing to the non-Abelian nature of the gauge group of general relativity which makes the procedure more complex than with the commutative gauge group of electromagnetism. In a treatment of the relationship between quantum theory and general relativity, Utiyama (1965) proposed the existence of a quantized gravitational analog of magnetic charge, but without exploring the ramifications of the suggestion. Dowker & Roche (1967) considered more thoroughly the same possibility, using linearized gravitational theory. Murai (1972) and Klimo & Dowker (1973) discussed magnetic monopoles within the context of general Yang–Mills fields. Recently Motz (1972) adapted Schwinger's line of reasoning to gravity and found as a consequence a quantum condition on mass in terms of the Planck mass $(\hbar c/G)^{1/2}$.

The derivation by Cabibbo & Ferrari of Dirac's quantum condition employs Mandelstam's path dependent formalism (1962a); these authors obtain the Dirac condition as a consistency requirement on the structure of path-dependent electromagnetic field theory. Their approach is essentially a relativistically covariant generalization of Dirac's original analysis, and as such it is particularly well suited to a similar application to general relativity. Therefore, in response to Motz, we attempted to show that the Cabibbo–Ferrari technique in gravitation does not lead to Motz's quantum condition, nor to any quantization at all (Riegert, 1974).

We now believe a deeper examination reveals that a quantum condition does occur for the sources of the gravitational field, but not the one Motz envisions; the basic argument in outline is simply presented.

The starting point of Cabibbo & Ferrari is a consideration of the gauge invariant derivative of a charged field χ:

$$D_\mu \chi = \chi_{,\mu} - (ieA_\mu \chi/\hbar) \tag{1.1}$$

where A_μ is the electromagnetic potential. Similarly, we may consider the covariant derivative of a spinor or tensor field ψ,

$$\psi_{;\mu} = \psi_{,\mu} + \Gamma_\mu \psi \tag{1.2}$$

where Γ_μ is the appropriate affinity. Note that whereas (1.1) contains Planck's constant in an essential way, (1.2) does not, since for ψ a tensor field, (1.2) is valid even in classical general relativistic field theory. Since the quantum conditions for the electromagnetic and gravitational fields are derived directly from (1.1) and (1.2), respectively, it follows that *the quantization condition on the source of the gravitational field does not involve $\hbar$.* Consequently, Motz's result cannot be realized.

It is easy to see, however, what alternative form the gravitational quantum condition must take by examining the electromagnetic case. The vanishing of the three-volume integral of the divergence of the dual Maxwell tensor in ordinary electrodynamics is replaced in monopole theory by

$$\int_V F^{*\mu\nu}{}_{,\nu}\, dV_\mu = Q_M$$

where Q_M is the quantized magnetic charge. By comparison, in general relativity we expect the three-volume integral of the divergence of the dual Riemann tensor, which normally vanishes by virtue of the Bianchi identity, to generalize to

$$\int_V R^*{}_{\alpha\beta}{}^{\mu\nu}{}_{;\nu}\, dV_\mu = Q_{\alpha\beta}$$

where some scalar formed from $Q_{\alpha\beta}$ is quantized.

Since the prime consequence of the Bianchi identity is the conservation of energy-momentum, in our generalized theory we might expect some sort of *quantized violation of local energy-momentum conservation.* Indeed, we have already remarked that Planck's constant cannot appear in our result, which leaves only the speed of light c and the gravitational constant G with which to express the quantization condition. We note that c^5/G has units of erg $\sec^{-1}$, as would be expected if the quantization involves the rate of production or destruction of energy.

It is perhaps necessary to emphasize that the nonoccurrence of $\hbar$ in our final result should not be taken as an indication that the quantum condition is a consequence of strictly classical considerations. Indeed, only within the framework of wave mechanics is the formalism of interacting fields sufficiently broad to embrace all physical phenomena, and only using field theory will we arrive at our desired goal.

In the following sections we attempt to make precise the foregoing conjectures.

2. *Coordinate Systems*

Select an arbitrary point O in space-time. By the equivalence principle, we may erect a global coordinate system x^μ (whose origin for convenience may be chosen to coincide with O) such that x^μ is locally inertial at O. In the language of the tetrad formalism (see, e.g., Kibble, 1961; and deWitt, 1965) this means that if $h_{i\mu}(x)$ is a field of orthonormal tetrads defined by

$$h_{i\mu}h^i{}_\nu = g_{\mu\nu} \tag{2.1a}$$

$$h_{i\mu}h_j{}^\mu = \eta_{ij} \tag{2.1b}$$

where

$$\eta_{ij} \equiv \mathrm{diag}(+1, -1, -1, -1) \tag{2.2}$$

then at the point O we have

$$h^i{}_\mu(0) = \delta^i{}_\mu \tag{2.3a}$$

$$h^i{}_{\mu,\sigma}(0) = 0 \tag{2.3b}$$

Note that here we are adopting the convention of employing the lower case Latin alphabet for locally inertial (Lorentz) coordinate frame indices and the lower case Greek alphabet for general curvilinear (global) coordinate system indices; both types of index run from 0 to 3.

Equation (2.3b) taken with (2.1a) implies that

$$g_{\mu\nu,\sigma}(0) = 0$$

which in turn forces the vanishing at O of the global affine connection [we are ignoring here the intriguing possibility of torsion (Kibble, 1961)]:

$$\Gamma^\lambda{}_{\mu\nu}(0) = 0 \tag{2.4}$$

The vanishing of the Christoffel symbol at O is of course necessary in any coordinate system which is inertial there.

The general covariant derivative of the tetrad is defined to vanish identically throughout space-time; that is,

$$0 = h^i{}_{\mu;\sigma} \equiv h^i{}_{\mu,\sigma} + A^i{}_{j\sigma} h^j{}_\mu - \Gamma^\lambda{}_{\mu\sigma} h^i{}_\lambda \tag{2.5}$$

Here, $A^i{}_{j\sigma}$ is the local affinity. As a consequence of (2.3b), (2.4), and (2.5), the local affinity also vanishes at O:

$$A^i{}_{j\sigma}(0) = 0 \tag{2.6}$$

The utility of equations (2.3), (2.4), and (2.6) lies in their use as boundary conditions on the solutions of certain differential equations to be considered in the sections to follow.

3. *Spinor Analysis*

One can obtain an understanding of the gravitational field not provided by the usual geometric approach of general relativity by considering gravitation, in analogy to electromagnetism, as a gauge field. In other words, we may consider gravity as the compensating field arising from the rather natural demand that the Lorentz covariant equations of special relativity should remain form invariant under the more general class of position-dependent Lorentz transformations. Proceeding in this fashion, we find that the tetrad formalism manifests itself in a direct manner (Kibble, 1961).

Moreover, one discovers in this way that the gauge group of gravidynamics is $SL(2, C)$, the six-parameter covering group of the homogeneous Lorentz group. (For further discussion of gravitation as a gauge field, including a possible generalization, the reader is referred to Salam, 1973.) Since $SL(2, C)$ contains $SU(2)$ as a subgroup, it is easy to see that spinors must enter essentially into any analysis of the gauge field aspects of gravitation. Indeed, since the appearance in nature of half-integral spin fields forces upon us consideration of the spinor representations of the Lorentz group, and because the entire Lorentz tensor formalism can easily be recast into spinor language, it is both necessary and convenient to discuss all representations of the homogeneous Lorentz group in terms of the spinor representations.

To this end, let ψ be a field that provides a matrix representation of the Lorentz group. Then the covariant derivative of ψ is defined by (DeWitt, 1965)

$$\psi_{|\mu} \equiv \psi_{,\mu} + \tfrac{1}{2} G^{ij} A_{ij\mu} \psi \tag{3.1}$$

where $G^{ij} = -G^{ji}$ are the generators of the representation and obey the commutation relations

$$[G^{ij}, G^{kl}] = \tfrac{1}{2} C^{ijkl}{}_{mn} G^{mn} \tag{3.2}$$

The $C^{ijkl}{}_{mn}$ are the structure constants of the group. We employ a vertical stroke for local covariant differentiation and a semicolon for general covariant differentiation. For purely local quantities, there is no difference between the two types of derivative.

It is convenient for future use to define in addition a global covariant derivative, indicated by a colon:

$$B_{i\mu:\nu} \equiv B_{i\mu,\nu} - \Gamma^{\lambda}{}_{\mu\nu} B_{i\lambda} \tag{3.3}$$

The general and global covariant derivatives of a strictly global tensor are identical.

The basic entity in spinor analysis is the fundamental spinor-tensor $\sigma^{iA\dot{B}}$. It satisfies

$$\sigma^{iA\dot{B}}\sigma^{j}{}_{\dot{B}C} + \sigma^{jA\dot{B}}\sigma^{i}{}_{\dot{B}C} = \eta^{ij}\,\delta^{A}_{C} \tag{3.4}$$

and is used to convert Lorentz tensor indices to spinor indices and vice versa. Here, and in what follows, we use upper case Latin letters for spinor indices, running from 1 to 2; in essential respects our spinor conventions follow those of Bade & Jehle (1953). The fundamental spinor–tensor is also by definition invariant under proper homogeneous Lorentz transformations; thus

$$\sigma^{iA}{}_{\dot{B}} = \sigma'^{iA}{}_{\dot{B}} \equiv \Lambda^{i}{}_{j} S^{A}{}_{C} \sigma^{jC}{}_{\dot{D}} S^{-1\dot{D}}{}_{\dot{B}} \tag{3.5}$$

where, for Lorentz transformations characterized by the six infinitesimal parameters $d\zeta^{ij} = -d\zeta^{ji}$,

$$\Lambda^{i}{}_{j} \equiv \delta^{i}_{j} + d\zeta^{i}{}_{j} \tag{3.6a}$$

$$S^{A}{}_{C} \equiv \delta^{A}_{C} + \tfrac{1}{2} G^{ijA}{}_{C}\, d\zeta_{ij} \tag{3.6b}$$

$$S^{-1\dot{D}}{}_{\dot{B}} \equiv \delta^{\dot{D}}_{\dot{B}} - \tfrac{1}{2} G^{ij\dot{D}}{}_{\dot{B}}\, d\zeta_{ij} \tag{3.6c}$$

Insertion of (3.6) into (3.5) yields to first order in $d\zeta^{ij}$ [after a straightforward but laborious manipulation which makes frequent use of (3.4)] the result

$$G^{ijA}{}_{C} = \tfrac{1}{2}(\sigma^{iA\dot{F}}\sigma^{j}{}_{\dot{F}C} - \sigma^{jA\dot{F}}\sigma^{i}{}_{\dot{F}C}) \tag{3.7a}$$

$$G^{ij\dot{D}}{}_{\dot{B}} = \tfrac{1}{2}(\sigma^{i\dot{D}E}\sigma^{j}{}_{E\dot{B}} - \sigma^{j\dot{D}E}\sigma^{i}{}_{E\dot{B}}) \tag{3.7b}$$

Observe that $\dot{G}^{ij} = \overline{G^{ij}}$, where the bar denotes complex conjugation. This is entirely expected since $\psi^{\dot{A}} \equiv \overline{\psi^{A}}$.

We may express the σ matrices in terms of the familiar Pauli spin matrices $\boldsymbol{\tau}$, which by definition satisfy

$$\tau_{\Delta}\tau_{\Lambda} = i\epsilon_{\Delta\Lambda\Sigma}\tau_{\Sigma} + \delta_{\Delta\Lambda} \tag{3.8}$$

Specifically,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.9}$$

Notice that we employ upper case Greek letters for three-vector indices, which take on the values 1, 2, 3. In (3.8), $\epsilon_{\Delta\Lambda\Sigma}$ is the three-dimensional Levi-Civita symbol, $\epsilon_{123} = +1$.

Defining a spin four-vector s^i by

$$s^i \equiv (1, \boldsymbol{\tau}) \tag{3.10}$$

we find (see Bade & Jehle, 1953)

$$\sigma^{iA\dot{B}} = (2)^{-1/2} s^i \tag{3.11a}$$

$$\sigma^i{}_{\dot{C}D} = (2)^{-1/2}(-1)^i \bar{s}^i = (2)^{-1/2}(1, -\boldsymbol{\tau}) \tag{3.11b}$$

Substituting (3.11) into (3.7) and applying (3.8) we have

$$G^{\Delta 0} = \tfrac{1}{2}\tau_\Delta, \qquad G^{\Delta\Lambda} = -\tfrac{1}{2} i\epsilon_{\Delta\Lambda\Sigma}\tau_\Sigma \tag{3.12a}$$

$$\dot{G}^{\Delta 0} = \tfrac{1}{2}\overline{\tau_\Delta}, \qquad \dot{G}^{\Delta\Lambda} = \tfrac{1}{2} i\epsilon_{\Delta\Lambda\Sigma}\overline{\tau_\Sigma} \tag{3.12b}$$

Equations (3.12) could have been arrived at via a perhaps more familiar route which starts by reexpressing the commutation relations (3.2) in terms of a pair of independent angular momentum matrix vectors J_Δ and K_Δ. In this way, we would discover that equations (3.12a) constitute the generators of the $(\frac{1}{2}, 0)$ representation, whereas (3.12b) generate the $(0, \frac{1}{2})$ representation (see, e.g., Schweber, 1961). The spinor formalism is better suited to our purpose, however. An arbitrary tensor or spinor transforms like some outer product of the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. This fact is accommodated in spinor algebra by the simple prescription of adding the appropriate numbers of undotted and dotted indices to the field variable. Therefore, without loss of generality, we may restrict our attention to the single index (spin-$\frac{1}{2}$) spinor representations.

As it bears on the main argument of this article, a still more cogent reason for taking as primary the half-integer spin representations is that consideration of a spin-$\frac{1}{2}$ field interacting with gravitation leads to the most restrictive quantization condition possible on the sources of the gravitational field, as may be seen by comparison to the model case of electrodynamics. That is, the treatment of a particle of charge e interacting with the electromagnetic field leads to a quantum condition on magnetic charge more restrictive than an identical treatment of a particle of charge αe, $\alpha > 1$, as is readily observed by substituting αe for e in the Dirac condition. A comparison of (1.1) and (3.1) demonstrates that, roughly speaking, analogous to the *charge* of a test particle in an electromagnetic field is the *spin* of a particle in a gravitational field. Hence, since $\frac{1}{2}$ is the smallest (nontrivial) spin a test particle may possess, analysis of a spin-$\frac{1}{2}$ field yields the most limiting quantization condition for gravity.

At this point, it would be wise to emphasize that, with respect to quantum field theory, all of the fields discussed herein should be considered as strictly classical entities. Thus, while the existence of spinor fields will be exploited, such fields are essentially quantum mechanical only within the context of first quantization, which makes it, if not plausible, at least possible to ignore completely the effects of second quantization.

Rewriting (3.1) for the $(\frac{1}{2}, 0)$ representation, we obtain

$$\psi^A{}_{|\mu} = \psi^A{}_{,\mu} + \tfrac{1}{2}G^{ijA}{}_C A_{ij\mu}\psi^C$$
$$= \psi^A{}_{,\mu} + (G^{\Delta 0 A}{}_C A_{\Delta 0\mu} + \tfrac{1}{2}G^{\Delta\Lambda A}{}_C A_{\Delta\Lambda\mu})\psi^C$$

or, suppressing spinor indices using matrix multiplication and inserting (3.12a),

$$\psi_{|\mu} = \psi_{,\mu} + \tfrac{1}{2}(\tau_\Delta A_{\Delta 0\mu} - \tfrac{1}{2}i\epsilon_{\Delta\Lambda\Sigma}\tau_\Sigma A_{\Delta\Lambda\mu})\psi$$
$$= \psi_{,\mu} + \tfrac{1}{2}\boldsymbol{\tau}\cdot \mathbf{A}_\mu\psi \tag{3.13}$$

where

$$\mathbf{A}_\mu \equiv A_{\Sigma\mu} \equiv A_{\Sigma 0\mu} - \tfrac{1}{2}i\epsilon_{\Delta\Lambda\Sigma}A_{\Delta\Lambda\mu} \tag{3.14}$$

A similar treatment of the $(0, \frac{1}{2})$ representation yields, in matrix notation (and with the obvious requirement that $A_{ij\mu}$ be real),

$$\dot{\psi}_{|\mu} = \dot{\psi}_{,\mu} + \tfrac{1}{2}\overline{\boldsymbol{\tau}\cdot \mathbf{A}_\mu}\dot{\psi} \tag{3.15}$$

Since (3.15) comes from (3.13) by the substitution $\boldsymbol{\tau}\cdot\mathbf{A}_\mu \to \overline{\boldsymbol{\tau}\cdot\mathbf{A}_\mu}$, all results applicable to the $(\frac{1}{2}, 0)$ (undotted) representation also apply after complex conjugation to the $(0, \frac{1}{2})$ (dotted) representation.

It may be demonstrated (DeWitt, 1965) that

$$\psi_{|\mu|\nu} - \psi_{|\nu|\mu} = \tfrac{1}{2}G^{ij}R_{ij\mu\nu}\psi \tag{3.16}$$

where

$$R_{ij\mu\nu} \equiv A_{ij\mu,\nu} - A_{ij\nu,\mu} + A_{ik\nu}A^k{}_{j\mu} - A_{ik\mu}A^k{}_{j\nu} \tag{3.17}$$

[when performing the second local differentiations on the left of (3.16), we simply neglect the presence of the global tensor indices.] Expressing the G^{ij} by (3.12a) and the $A_{ij\mu}$ by (3.14), (3.16) and (3.17) become

$$\psi_{|\mu|\nu} - \psi_{|\nu|\mu} = \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{\mu\nu}\psi \tag{3.18}$$

$$\mathbf{R}_{\mu\nu} = \mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} + i(\mathbf{A}_\nu \times \mathbf{A}_\mu) \tag{3.19}$$

4. *The Path Dependence Formalism*

We introduce a matrix $W(x, P)$ by requiring that

$$\partial_\mu W(x, P) = \tfrac{1}{2}W(x, P)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(x) \tag{4.1}$$

In view of (3.13), this implies

$$\partial_\mu[W(x, P)\psi(x)] = W(x, P)[\psi(x)_{|\mu}] \tag{4.2}$$

The P in $W(x, P)$ denotes path dependence, as may be seen by integrating (4.1):

$$W(x, P) = k + \frac{1}{2}\int_P^x W\boldsymbol{\tau}\cdot\mathbf{A}_\mu\, d\xi^\mu$$

The integration is taken from O, the origin of the global coordinate system, along the path P to the point x^μ. We may fix the constant k by the requirement

$$\Psi(0,P) \equiv W(0,P)\psi(0) = \psi(0)$$

This is reasonable, since the path dependence of a quantity reflects its interaction with the gravitational field; in the absence of gravitation there should exist no difference between the path-dependent field Ψ and the ordinary field ψ. Since (2.3a), (2.4), and (2.6) tell us that in the vicinity of O all effects of gravitation have been eliminated, the result $W(0,P) = 1$ follows. Thus,

$$W(x,P) = 1 + \int_P^x W\boldsymbol{\tau}\cdot \mathbf{A}_\mu \, d\xi^\mu \tag{4.3}$$

Using (4.1) and the identity

$$0 = \partial_\mu(WW^{-1}) = (\partial_\mu W)W^{-1} + W\,\partial_\mu W^{-1}$$

we have in addition

$$\partial_\mu W^{-1}(x,P) = -\tfrac{1}{2}\boldsymbol{\tau}\cdot \mathbf{A}_\mu(x) W^{-1}(x,P) \tag{4.4a}$$

$$W^{-1}(x,P) = 1 - \frac{1}{2}\int_P^x \boldsymbol{\tau}\cdot \mathbf{A}_\mu W^{-1}\, d\xi^\mu \tag{4.4b}$$

In order to raise and lower spinor indices, we may introduce into spinor space a metric γ defined by (Bade and Jehle, 1953)

$$\gamma_{AB} \equiv \gamma_{\dot{A}\dot{B}} \equiv \gamma^{AB} \equiv \gamma^{\dot{A}\dot{B}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then, $\psi^A \gamma_{BA} W^{-1B}{}_C \equiv \psi_B W^{-1B}{}_C$ has properties exactly analogous to $W^A{}_B \psi^B$:

$$\partial_\mu(\psi_B W^{-1B}{}_C) = \psi_{B|\mu} W^{-1B}{}_C$$

In general, with the four basic types of spinors (contravariant and covariant, dotted and undotted) ψ^A, ψ_A, $\psi^{\dot{B}}$, $\psi_{\dot{B}}$, we associate the four path-dependent matrices $W^C{}_D$, $W^{-1C}{}_D$, $W^{\dot{E}}{}_{\dot{F}}$, $W^{-1\dot{E}}{}_{\dot{F}}$, respectively. Because of the formal similarities between all four entities, we shall for the most part deal only with undotted, contravariant spinors. The equations obtained may be used for the other spinors by application of the metric γ and/or complex conjugation.

Under a space-time-dependent Lorentz transformation $S(x)$ which satisfies the boundary condition $S(0) = 1$, we have

$$\psi' \equiv S\psi \tag{4.5a}$$

$$\tfrac{1}{2}\boldsymbol{\tau}\cdot \mathbf{A}'_\mu = \tfrac{1}{2}S\boldsymbol{\tau}\cdot \mathbf{A}_\mu S^{-1} - (\partial_\mu S)S^{-1} \tag{4.5b}$$

where the second equation is obtained by assuming the covariant derivative of ψ transforms as a spinor; the second term on the right of (4.5b) arises because $A_{jj\mu}$ transforms affinely under the Lorentz group. Now, by (4.1),

$$\partial_\mu W'(x,P) = \tfrac{1}{2} W' \boldsymbol{\tau} \cdot \mathbf{A}'_\mu \tag{4.6a}$$

Similarly,

$$\begin{aligned}
\partial_\mu [W(x,P)S^{-1}(x)] &= (\partial_\mu W)S^{-1} + W\partial_\mu S^{-1} \\
&= \tfrac{1}{2} W\boldsymbol{\tau}\cdot \mathbf{A}_\mu S^{-1} + W\partial_\mu S^{-1} \\
&= WS^{-1}(\tfrac{1}{2} S\boldsymbol{\tau}\cdot \mathbf{A}_\mu S^{-1} + S\,\partial_\mu S^{-1}) \\
&= WS^{-1}[\tfrac{1}{2} S\boldsymbol{\tau}\cdot \mathbf{A}_\mu S^{-1} - (\partial_\mu S)S^{-1}] \\
&= \tfrac{1}{2} WS^{-1}\boldsymbol{\tau}\cdot \mathbf{A}'_\mu
\end{aligned} \tag{4.6b}$$

where in the final step use has been made of (4.5b). Comparing (4.6a) and (4.6b), and recalling $W(0,P) = W'(0,P) = 1$, we find $W' = WS^{-1}$, or

$$W = W'S \tag{4.7}$$

Making the natural definition

$$\Psi(x,P) \equiv W(x,P)\psi(x) \tag{4.8}$$

we observe that, by virtue of (4.5a) and (4.7),

$$\Psi'(x,P) \equiv W'\psi' = W'S\psi = W\psi = \Psi(x,P)$$

Therefore, Ψ is path dependent but independent of Lorentz transformations which may vary from point to point in space-time.

Define $M(x, y, \Delta P)$ by

$$W(y, P + \Delta P) = M(x, y, \Delta P)W(x, P) \tag{4.9}$$

where ΔP is the path segment connecting points x and y. Obviously, (4.9) is equivalent to the equations (suppressing the explicit path dependence)

$$M(x,y) = W(y)W^{-1}(x) \tag{4.10a}$$

$$W^{-1}(x) = W^{-1}(y)M(x,y) \tag{4.10b}$$

Differentiating $M(x, y)$ with respect to each of its two arguments,

$$\begin{aligned}
\partial M(x,y)/\partial y^\mu &= \partial[W(y)W^{-1}(x)]/\partial y^\mu \\
&= [\partial W(y)/\partial y^\mu]W^{-1}(x) \\
&= \tfrac{1}{2} W(y)\boldsymbol{\tau}\cdot \mathbf{A}_\mu(y)W^{-1}(x) \\
&= \tfrac{1}{2} W(y)\boldsymbol{\tau}\cdot \mathbf{A}_\mu(y)W^{-1}(y)M(x,y)
\end{aligned} \tag{4.11a}$$

and

$$\begin{aligned}\partial M(x,y)/\partial x^\mu &= \partial\,[W(y)W^{-1}(x)]/\partial x^\mu \\ &= W(y)\,\partial W^{-1}(x)/\partial x^\mu \\ &= -\tfrac{1}{2}W(y)\,\boldsymbol{\tau}\cdot\mathbf{A}_\mu(x)W^{-1}(x) \\ &= -\tfrac{1}{2}M(x,y)W(x)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(x)W^{-1}(x) \qquad (4.11\text{b})\end{aligned}$$

In obtaining (4.11), we use (4.1), (4.4a), (4.9), and (4.10).

We now demonstrate that (4.11a) and (4.11b), together with the necessary condition that $M(x,x) = 1$ when $\Delta P = 0$, are satisfied by

$$M(x,y) = 1 + \frac{1}{2}\int_x^y M(\xi,y)W(\xi)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(\xi)W^{-1}(\xi)\,d\xi^\mu \qquad (4.12)$$

where the integral is evaluated along ΔP. To prove this, observe that according to (4.12)

$$\partial M(x,y)/\partial x^\mu = -\tfrac{1}{2}M(x,y)W(x)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(x)W^{-1}(x)$$

which is identical to (4.11b). Also, by (4.12)

$$\begin{aligned}\partial M(x,y)/\partial y^\mu &= \tfrac{1}{2}M(y,y)W(y)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(y)W^{-1}(y) \\ &\quad + \frac{1}{2}\int_x^y [\partial M(\xi,y)/\partial y^\mu]\,W(\xi)\boldsymbol{\tau}\cdot\mathbf{A}_\alpha(\xi)W^{-1}(\xi)\,d\xi^\alpha\end{aligned}$$

Inserting the value 1 for $M(y,y)$ and substituting the desired expression (4.11a) for $\partial M(\xi,y)/\partial y^\mu$, we obtain

$$\begin{aligned}\partial M(x,y)/\partial y^\mu &= \tfrac{1}{2}W(y)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(y)W^{-1}(y) \\ &\quad + \frac{1}{2}\int_x^y [\tfrac{1}{2}W(y)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(y)W^{-1}(y)M(\xi,y)]\,W(\xi)\boldsymbol{\tau}\cdot\mathbf{A}_\alpha(\xi)W^{-1}(\xi)\,d\xi^\alpha \\ &= \tfrac{1}{2}W(y)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(y)W^{-1}(y)\left[1 + \frac{1}{2}\int_x^y M(\xi,y)W(\xi)\boldsymbol{\tau}\cdot\mathbf{A}_\alpha(\xi)W^{-1}(\xi)\,d\xi^\alpha\right] \\ &= \tfrac{1}{2}W(y)\boldsymbol{\tau}\cdot\mathbf{A}_\mu(y)W^{-1}(y)M(x,y)\end{aligned}$$

which is (4.11a). The validity of (4.12) is thus established. Reference to (4.9) shows that (4.12) may be rewritten as

$$M(x,y,\Delta P) = 1 + \tfrac{1}{2}W(y)\int_x^y \boldsymbol{\tau}\cdot\mathbf{A}_\mu(\xi)W^{-1}(\xi)\,d\xi^\alpha \qquad (4.13)$$

In (4.13), if $x = y$, but the point is approached by two different paths so that ΔP is a nonzero, closed curve, then

$$M(x, \Delta P) = 1 + \tfrac{1}{2}W(x) \oint \boldsymbol{\tau}\cdot \mathbf{A}_\mu(\xi)W^{-1}(\xi)\, d\xi^\mu$$

$$= 1 + \tfrac{1}{4}W(x) \int_\Sigma \boldsymbol{\tau}\cdot [\partial(\mathbf{A}_\mu W^{-1})/\partial\xi^\nu - \partial(\mathbf{A}_\nu W^{-1})/\partial\xi^\mu]\, dS^{\mu\nu}(\xi), \quad (4.14)$$

by Stokes' theorem, where Σ is a surface bounded by ΔP. With regard to our integral theorem conventions, we adhere basically to the treatment of Landau & Lifshitz (1971).

We have

$$\partial_\nu(\mathbf{A}_\mu W^{-1}) = (\partial_\nu \mathbf{A}_\mu)W^{-1} + \mathbf{A}_\mu\, \partial_\nu W^{-1}$$
$$= (\mathbf{A}_{\mu,\nu} - \tfrac{1}{2}\mathbf{A}_\mu \boldsymbol{\tau}\cdot \mathbf{A}_\nu)W^{-1}$$

Therefore,

$$\partial_\nu(\mathbf{A}_\mu W^{-1}) - \partial_\mu(\mathbf{A}_\nu W^{-1}) = (\mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu}$$
$$- \tfrac{1}{2}\mathbf{A}_\mu\boldsymbol{\tau}\cdot \mathbf{A}_\nu + \tfrac{1}{2}\mathbf{A}_\nu\boldsymbol{\tau}\cdot \mathbf{A}_\mu)W^{-1}$$

Since by (3.8), (3.19), and the above

$$\boldsymbol{\tau}\cdot [\partial_\nu(\mathbf{A}_\mu W^{-1}) - \partial_\mu(\mathbf{A}_\nu W^{-1})] = \boldsymbol{\tau}\cdot (\mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu})W^{-1}$$
$$+ \tfrac{1}{2}[-\mathbf{A}_\mu\cdot \mathbf{A}_\nu - i(\mathbf{A}_\mu \times \mathbf{A}_\nu)\cdot\boldsymbol{\tau} + \mathbf{A}_\nu\cdot \mathbf{A}_\mu + i(\mathbf{A}_\nu \times \mathbf{A}_\mu)\cdot\boldsymbol{\tau}]\, W^{-1}$$
$$= \boldsymbol{\tau}\cdot [\mathbf{A}_{\mu,\nu} - \mathbf{A}_{\nu,\mu} + i(\mathbf{A}_\nu \times \mathbf{A}_\mu)]W^{-1}$$
$$= \boldsymbol{\tau}\cdot \mathbf{R}_{\mu\nu}W^{-1}$$

(4.14) becomes

$$M(\Sigma) = 1 + \tfrac{1}{4}W(x) \int_\Sigma \boldsymbol{\tau}\cdot\mathbf{R}_{\mu\nu}(\xi)W^{-1}(\xi)\, dS^{\mu\nu}(\xi)$$

$$= 1 + \tfrac{1}{4} \int_\Sigma M(\xi, x)W(\xi)\boldsymbol{\tau}\cdot \mathbf{R}_{\mu\nu}(\xi)W^{-1}(\xi)\, dS^{\mu\nu}(\xi) \quad (4.15)$$

Note that while it would appear that the right-hand side of (4.15) includes x dependence, this conclusion would be erroneous since according to (4.10a)

$$\partial_\mu M(x, \Sigma) = [\partial_\mu W(x, P')]\, W^{-1}(x, P) + W(x, P')[\partial_\mu W^{-1}(x, P)]$$
$$= [\tfrac{1}{2}W(x, P')\boldsymbol{\tau}\cdot \mathbf{A}_\mu(x)]\, W^{-1}(x, P) + W(x, P')[-\tfrac{1}{2}\boldsymbol{\tau}\cdot \mathbf{A}_\mu(x)W^{-1}(x, P)]$$
$$= 0$$

Consider the derivative of $W\boldsymbol{\tau}W^{-1}$:

$$\partial_\alpha(W\boldsymbol{\tau}W^{-1}) = (\partial_\alpha W)\boldsymbol{\tau}W^{-1} + W\boldsymbol{\tau}\,\partial_\alpha W^{-1}$$
$$= \tfrac{1}{2}W\boldsymbol{\tau}\cdot \mathbf{A}_\alpha\boldsymbol{\tau}W^{-1} - \tfrac{1}{2}W\boldsymbol{\tau}\boldsymbol{\tau}\cdot \mathbf{A}_\alpha W^{-1}$$

Thus, by (3.8)

$$\begin{aligned}\partial_\alpha(W\boldsymbol{\tau}W^{-1}\cdot\mathbf{R}_{\mu\nu}) &= \tfrac{1}{2}W\boldsymbol{\tau}\cdot\mathbf{A}_\alpha\boldsymbol{\tau}W^{-1}\cdot\mathbf{R}_{\mu\nu} - \tfrac{1}{2}W\boldsymbol{\tau}\cdot\mathbf{R}_{\mu\nu}\boldsymbol{\tau}\cdot\mathbf{A}_\alpha W^{-1} \\ &\quad + W\boldsymbol{\tau}W^{-1}\cdot\partial_\alpha\mathbf{R}_{\mu\nu} \\ &= \tfrac{1}{2}W[\mathbf{A}_\alpha\cdot\mathbf{R}_{\mu\nu} + i(\mathbf{A}_\alpha\times\mathbf{R}_{\mu\nu})\cdot\boldsymbol{\tau} - \mathbf{R}_{\mu\nu}\cdot\mathbf{A}_\alpha - i(\mathbf{R}_{\mu\nu}\times\mathbf{A}_\alpha)\cdot\boldsymbol{\tau}]W^{-1} \\ &\quad + W\boldsymbol{\tau}W^{-1}\cdot\partial_\alpha\mathbf{R}_{\mu\nu} \\ &= W\boldsymbol{\tau}W^{-1}\cdot[\mathbf{R}_{\mu\nu,\alpha} + i(\mathbf{A}_\alpha\times\mathbf{R}_{\mu\nu})]\end{aligned} \tag{4.16}$$

It can easily be shown by expansion in terms of the definitions (3.14) and (3.19) that the local covariant derivative of the curvature tensor

$$R_{ij\mu\nu|\alpha} \equiv R_{ij\mu\nu,\alpha} - A^k{}_{i\alpha}R_{kj\mu\nu} - A^k{}_{j\alpha}R_{ik\mu\nu}$$

may be equivalently expressed as

$$\mathbf{R}_{\mu\nu|\alpha} = \mathbf{R}_{\mu\nu,\alpha} + i(\mathbf{A}_\alpha\times\mathbf{R}_{\mu\nu})$$

With this result (4.16) becomes

$$\partial_\alpha(W\boldsymbol{\tau}W^{-1}\cdot\mathbf{R}_{\mu\nu}) = W\boldsymbol{\tau}W^{-1}\cdot(\mathbf{R}_{\mu\nu|\alpha}) \tag{4.17}$$

Let us introduce the quantity $V_{\Lambda\Sigma}$ by

$$\partial_\alpha(V_{\Lambda\Sigma}R_{\Sigma\mu\nu}) = V_{\Lambda\Sigma}(R_{\Sigma\mu\nu|\alpha}) \tag{4.18}$$

That is, by analogy to (4.8),

$$\mathbf{R}_{P\mu\nu} \equiv R_{P\Lambda\mu\nu} \equiv V_{\Lambda\Sigma}R_{\Sigma\mu\nu}$$

is the path-dependent, Lorentz-transformation-independent gravitational field tensor. Comparing (4.17) and (4.18), it is not difficult to see that

$$\tau_\Lambda V_{\Lambda\Sigma}R_{\Sigma\mu\nu} \equiv \boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu} = W\boldsymbol{\tau}W^{-1}\cdot\mathbf{R}_{\mu\nu}$$

Therefore, (4.15) becomes

$$M(\Sigma) = 1 + \frac{1}{4}\int_\Sigma M(\xi, x)\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu}(\xi)\,dS^{\mu\nu}(\xi) \tag{4.19}$$

Combining (4.8) with (4.9),

$$W(y, P+\Delta P)\psi(x) = M(x, y, \Delta P)\Psi(x, P)$$

or, assuming $x = y$ but $\Delta P \neq 0$,

$$\Psi(x, P') = M(\Sigma)\Psi(x, P) \tag{4.20}$$

where $M(\Sigma)$ is given by (4.19). Equation (4.20) represents the generalization to finite closed curves of the well-known result of parallel transport around an infinitesimal loop in curved space; we mention in passing that (4.20) results in the gravitational analog of the Aharonov–Bohm effect (1959).

Equation (4.20) also demonstrates the importance of the condition $\partial_\mu M(\Sigma) = 0$; since this relation is essential according to (4.20) if *derivatives* of a path dependent quantity are to have the same path dependence properties as the quantity itself. This is exactly similar to the situation in normal gauge invariant theories where the gauge invariant derivative of a gauge-dependent quantity is defined so as to transform the same as the quantity itself. Moreover, (4.20) and the vanishing of the ordinary derivative of $M(\Sigma)$ ensures that if $\Psi(x, P)$ satisfies some differential field equation then so also will $\Psi(x, P')$.

According to (4.19) and (4.20), if P' differs from P only infinitesimally, then

$$\Psi(x, P') = [1 + \tfrac{1}{4}M(x, x)\boldsymbol{\tau}\cdot \mathbf{R}_{P\mu\nu}(x)\, dS^{\mu\nu}]\, \Psi(x, P)$$

where $dS^{\mu\nu}$ is the surface element between P and P'. Since $M(x, x) = 1$, this becomes

$$\Psi(x, P') - \Psi(x, P) = \tfrac{1}{4}\boldsymbol{\tau}\cdot \mathbf{R}_{P\mu\nu}(x)\, dS^{\mu\nu}\Psi(x, P) \tag{4.21}$$

5. *The Consistency Condition*

The surface of integration Σ appearing in (4.20) is bounded by ΔP, the closed path connecting the point x to itself, but Σ is otherwise arbitrary as there is no apparent criterion by which to choose one surface of integration over any other. Thus, if Σ and Σ' are two distinct surfaces both bounded by the same closed curve ΔP, the internal consistency of the path dependency approach to field theory requires that $M(\Sigma) = M(\Sigma')$. In particular, let $\Sigma \to 0$. Then evidently $\Sigma' \to \Sigma_c$, a closed surface containing three-volume V. As $\Sigma \to 0$, we must demand that $M(\Sigma) \to 1$, so $M(\Sigma') \to 1$ also, or

$$M(\Sigma_c) = 1 \tag{5.1}$$

Combining (5.1) with (4.19),

$$1 = 1 + \frac{1}{4}\int_{\Sigma_c} M\boldsymbol{\tau}\cdot \mathbf{R}_{P\mu\nu}\, dS^{\mu\nu}$$

or

$$\int_{\Sigma_c} M\boldsymbol{\tau}\cdot \mathbf{R}_{P\mu\nu}\, dS^{\mu\nu} = 0 \tag{5.2}$$

the three-volume V contained inside Σ_c in (5.1) and (5.2) may take on any value since the selection of the surfaces Σ and Σ' was arbitrary. Requirement (5.2) is not as restrictive as the stronger condition

$$\int_{\Sigma_c} \boldsymbol{\tau}\cdot \mathbf{R}_{P\mu\nu}\, dS^{\mu\nu} = 0$$

which would lead, by Gauss' theorem, to the Bianchi identity of normal general relativity.

Define

$$\boldsymbol{\rho} \equiv \frac{1}{4} \int_{\Sigma} \mathbf{R}_{P\mu\nu}\, dS^{\mu\nu} \tag{5.3}$$

so that

$$\partial \boldsymbol{\rho} / \partial S^{\mu\nu} = \tfrac{1}{4} \mathbf{R}_{P\mu\nu} \tag{5.4}$$

Then, by virtue of (3.8), (4.19), (5.3), (5.4), and the well-known identity

$$\exp(\pm \boldsymbol{\tau} \cdot \boldsymbol{\rho}) = \cosh \rho \pm [(\boldsymbol{\tau} \cdot \boldsymbol{\rho})/\rho] \sinh \rho \tag{5.5a}$$

$$\rho \equiv \| \boldsymbol{\rho} \| \equiv (\boldsymbol{\rho} \cdot \boldsymbol{\rho})^{1/2} \tag{5.5b}$$

there holds the relation

$$M(\Sigma) = \left\{ 1 + \int_{\Sigma} M \left[\left(\cosh \rho - \frac{\sinh \rho}{\rho} \right) \left(\tfrac{1}{4} \mathbf{R}_{P\mu\nu} - \frac{\boldsymbol{\rho}}{\rho} \frac{\partial \rho}{\partial S^{\mu\nu}} \right) + \frac{i \sinh \rho}{4\rho} (\boldsymbol{\rho} \times \mathbf{R}_{P\mu\nu}) \right] \cdot \boldsymbol{\tau}\, dS^{\mu\nu} \right\} \exp(\boldsymbol{\tau} \cdot \boldsymbol{\rho}) \tag{5.6}$$

This may be verified (see the Appendix) by differentiating both sides of the equation with respect to $S^{\mu\nu}$ and recalling $M(\Sigma = 0) = 1$.

The identity (5.6) applied to a closed surface of integration leads directly to the quantization condition. Thus, letting $\Sigma = \Sigma_c$ in (5.6) and remembering (5.1)

$$\exp(-\boldsymbol{\tau} \cdot \boldsymbol{\rho}_c) = 1 + \int_{\Sigma_c} M \left[\left(\cosh \rho - \frac{\sinh \rho}{\rho} \right) \left(\frac{1}{4} \mathbf{R}_{P\mu\nu} - \frac{\boldsymbol{\rho}}{\rho} \frac{\partial \rho}{\partial S^{\mu\nu}} \right) + \frac{i \sinh \rho}{4\rho} (\boldsymbol{\rho} \times \mathbf{R}_{P\mu\nu}) \right] \cdot \boldsymbol{\tau}\, dS^{\mu\nu} \tag{5.7}$$

where

$$\boldsymbol{\rho}_c \equiv \boldsymbol{\rho}(\Sigma_c) \equiv \frac{1}{4} \int_{\Sigma_c} \mathbf{R}_{P\mu\nu}\, dS^{\mu\nu} \tag{5.8}$$

An application of Gauss' theorem transforms (5.8) into the more useful form

$$\boldsymbol{\rho}_c = \frac{1}{4} \int_{V} \epsilon^{\sigma\mu\nu\tau} \mathbf{R}_{P\mu\nu,\tau}\, dV_\sigma \tag{5.9}$$

where V is again the arbitrary three-volume enclosed by Σ_c and $\epsilon^{\sigma\mu\nu\tau}$ is the four-dimensional global Levi-Civita tensor, defined in relation to the flat-space Levi-Civita tensor e^{ijkl}, $e^{0123} = +1$, by

$$\epsilon^{\sigma\mu\nu\tau} = h_i{}^\sigma h_j{}^\mu h_k{}^\nu h_l{}^\tau e^{ijkl}$$

$$= h e^{\sigma\mu\nu\tau}$$

where

$$h \equiv \det(h_a{}^\alpha)$$

Because of the symmetry of $\Gamma^\lambda{}_{\alpha\beta}$ in its covariant indices, it is easy to show that

$$\tfrac{1}{2}\epsilon^{\sigma\mu\nu\tau}\mathbf{R}_{\mu\nu|\tau} = \tfrac{1}{2}\epsilon^{\sigma\mu\nu\tau}\mathbf{R}_{\mu\nu;\tau}$$

Since $\epsilon^{\sigma\mu\nu\tau}$ has a vanishing covariant derivative, this becomes

$$\tfrac{1}{2}\epsilon^{\sigma\mu\nu\tau}\mathbf{R}_{\mu\nu|\tau} = \tfrac{1}{2}(\epsilon^{\sigma\mu\nu\tau}\mathbf{R}_{\mu\nu})_{;\tau} = \mathbf{R}^{*\sigma\tau}{}_{;\tau}$$

where the last form is by definition of the right dual curvature tensor. The preceding is also valid for the path-dependent Riemann tensor, with the proviso that the general covariant derivative be replaced by the global covariant derivative defined by (3.3):

$$\tfrac{1}{2}\epsilon^{\sigma\mu\nu\tau}\mathbf{R}_{P\mu\nu,\tau} = \mathbf{R}_P^{*\sigma\tau}{}_{:\tau}$$

Therefore, (5.9) is expressible as

$$\boldsymbol{\rho}_c = \frac{1}{2}\int_V \mathbf{R}_P^{*\sigma\tau}{}_{:\tau}\, dV_\sigma \tag{5.10}$$

If the Bianchi identity

$$\mathbf{R}_P^{*\sigma\tau}{}_{:\tau} = 0$$

holds throughout V, (5.10) forces $\boldsymbol{\rho}_c$ to vanish and vice versa (because V is arbitrary). Since the Bianchi identity (and consequently normal Riemann spacetime) should represent one possible solution of (5.7), $\boldsymbol{\rho}_c = 0$ must satisfy (5.7), which in turn requires that

$$\int_{\Sigma_C} M\left[\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\left(\frac{1}{4}\mathbf{R}_{P\mu\nu} - \frac{\boldsymbol{\rho}}{\rho}\,\frac{\partial\rho}{\partial S^{\mu\nu}}\right) + \frac{i\sinh\rho}{4\rho}(\boldsymbol{\rho}\times\mathbf{R}_{P\mu\nu})\right]\cdot\boldsymbol{\tau}\, dS^{\mu\nu} = 0$$

Given this, for nonzero $\boldsymbol{\rho}_c$ (5.7) becomes

$$\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho}_c) = 1$$

The quantum condition therefore takes the form

$$\rho_c = \left\|\frac{1}{4}\int_{\Sigma_C}\mathbf{R}_{P\mu\nu}\, dS^{\mu\nu}\right\| = \left\|\frac{1}{2}\int_V \mathbf{R}_P^{*\sigma\tau}{}_{:\tau}\, dV_\sigma\right\| = 2\pi i n \tag{5.11a}$$

$$n = 0, \pm 1, \pm 2, \ldots \tag{5.11b}$$

Equation (5.11) is similar to a quantization condition on the gravitational field first proposed by Utiyama (1965) on the basis of somewhat different reasoning.

Notice that according to (5.11), ρ_c is discontinuous at those regions in space where ρ_c changes from quantum number n to n'. Since it is disallowed by

(5.11) that ρ_c should change continuously as Σ_c is deformed or shrunk in a continuous manner, these regions of discontinuity must occur only at isolated points. Because the ordinary theory of the gravitational field as a consequence of the Bianchi identity takes $\rho_c = 0$ and is in full accord with experience, we conclude that ρ_c *is everywhere zero except possibly at isolated points in space; at these points the condition (5.11) must hold.* In other words, the second integrand in (5.11a), $\mathbf{R}_P^{*\sigma\tau}{:}_\tau$, must have the form of a sum of three-space δ functionals normalized in accordance with (5.11). *The sources of the dual Riemann tensor are quantized and point-like.*

Squaring (5.11a) provides

$$\left(\int_V \mathbf{R}_P^{*\sigma\tau}{:}_\tau \, dV_\sigma\right)\cdot\left(\int_V \mathbf{R}_P^{*\alpha\beta}{:}_\beta dV_\alpha\right) = -16\pi^2 n^2 \tag{5.12}$$

Recalling the definition of $\mathbf{R}_{P\mu\nu}$ and the form of the antisymmetric four-tensor to complex three-vector mapping given by (3.14), it may be shown by direct expansion that

$$R_{P\Delta\mu\nu} \equiv V_{\Delta\Sigma} R_{\Sigma\mu\nu} = (-\tfrac{1}{2} V_{ij}{}^{kl} - \tfrac{1}{4} i e^{kl}{}_{mn} V_{ij}{}^{mn}) R_{kl\mu\nu}$$

Using this with (4.18), we obtain

$$R_{Pij\mu\nu,\sigma} = (-\tfrac{1}{2} V_{ij}{}^{kl} - \tfrac{1}{4} i e^{kl}{}_{mn} V_{ij}{}^{mn})(R_{kl\mu\nu|\sigma}) \tag{5.13}$$

Let us investigate the derivative of the quantity $\sigma_{iA}{}^{\dot B} W^A{}_C \sigma^{jC}{}_{\dot D} W^{-1\dot D}{}_{\dot B}$:

$$\begin{aligned}
\partial_\mu(\sigma_{iA}{}^{\dot B} W^A{}_C \sigma^{jC}{}_{\dot D} W^{-1\dot D}{}_{\dot B}) &= \sigma_{iA}{}^{\dot B} \sigma^{jC}{}_{\dot D} [(\partial_\mu W^A{}_C) W^{-1\dot D}{}_{\dot B} \\
&\quad + W^A{}_C (\partial_\mu W^{-1\dot D}{}_{\dot B})] \\
&= \sigma_{iA}{}^{\dot B} \sigma^{jC}{}_{\dot D} (\tfrac{1}{2} W^A{}_E \boldsymbol{\tau}^E{}_C \cdot \mathbf{A}_\mu W^{-1\dot D}{}_{\dot B} - \tfrac{1}{2} W^A{}_C \boldsymbol{\tau}^{\dot D}{}_{\dot F} \cdot \overline{\mathbf{A}_\mu} W^{-1\dot F}{}_{\dot B}) \\
&= \sigma_{iA}{}^{\dot B} W^A{}_C [\tfrac{1}{2} \boldsymbol{\tau}^C{}_E \cdot \mathbf{A}_\mu \sigma^{jE}{}_{\dot D} - \tfrac{1}{2} \boldsymbol{\tau}^{\dot F}{}_{\dot D} \cdot \overline{\mathbf{A}_\mu} \sigma^{jC}{}_{\dot F}] W^{-1\dot D}{}_{\dot B}
\end{aligned} \tag{5.14}$$

The local covariant derivative of the fundamental spinor-tensor σ vanishes identically (Bade and Jehle, 1953):

$$0 = \sigma^{jC}{}_{\dot D|\mu} \equiv A^j{}_{k\mu} \sigma^{kC}{}_{\dot D} + \tfrac{1}{2} \boldsymbol{\tau}^C{}_E . \mathbf{A}_\mu \sigma^{jE}{}_{\dot D} - \tfrac{1}{2} \boldsymbol{\tau}^{\dot F}{}_{\dot D} \cdot \overline{\mathbf{A}_\mu} \sigma^{jC}{}_{\dot F} \tag{5.15}$$

This result is of course a consequence of (3.5), the invariance of σ under Lorentz transformations; this is a practical convenience, since the conversion of tensor indices to spinor indices and conversely thereby commutes with covariant differentiation. From (5.15),

$$-A^j{}_{k\mu} \sigma^{kC}{}_{\dot D} = \tfrac{1}{2} \boldsymbol{\tau}^C{}_E \cdot \mathbf{A}_\mu \sigma^{jE}{}_{\dot D} - \tfrac{1}{2} \boldsymbol{\tau}^{\dot F}{}_{\dot D} \cdot \overline{\mathbf{A}_\mu} \sigma^{jC}{}_{\dot F}$$

Inserting this into (5.14) yields

$$\begin{aligned}
\partial_\mu(\sigma_{iA}{}^{\dot B} W^A{}_C \sigma^{jC}{}_{\dot D} W^{-1\dot D}{}_{\dot B}) &= -\sigma_{iA}{}^{\dot B} W^A{}_C A^j{}_{k\mu} \sigma^{kC}{}_{\dot D} W^{-1\dot D}{}_{\dot B} \\
&= -\sigma_{iA}{}^{\dot B} W^A{}_C \sigma^{kC}{}_{\dot D} W^{-1\dot D}{}_{\dot B} A^j{}_{k\mu}
\end{aligned} \tag{5.16}$$

Define

$$U_i{}^j(x,P) \equiv \sigma_{iA}{}^{\dot{B}} W^A{}_C \sigma^{jC}{}_{\dot{D}} W^{-1\dot{D}}{}_{\dot{B}} \tag{5.17}$$

so that (5.16) becomes

$$\partial_\mu U_i{}^j = -U_i{}^k A^j{}_{k\mu} \tag{5.18}$$

it is apparent that $U_i{}^j(x,P)$ is the tensor analog of $W^A{}_B(x,P)$. We may use $U_i{}^j$ to consider path-dependent, generalized Lorentz invariant four-vectors of the sort J_{Pi}. That is,

$$\begin{aligned} J_{Pi,\mu} &= (U_i{}^j J_j)_{,\mu} = U_i{}^j (J_{j,\mu} - A^k{}_{j\mu} J_k) \\ &= U_i{}^j (J_{j|\mu}) \end{aligned}$$

According to (5.17), employing $U_i{}^j$ is a handy shorthand for the procedure of converting local tensor indices to spinor indices with σ, then changing to path dependence, generalized Lorentz invariance by use of the W matrices, and finally converting back to tensor indices by reapplying σ.

In particular,

$$R_{Pij\mu\nu,\sigma} \equiv (U_i{}^k U_j{}^l R_{kl\mu\nu})_{,\sigma} = U_i{}^k U_j{}^l (R_{kl\mu\nu|\sigma}) \tag{5.19}$$

Comparison of (5.13) with (5.19) reveals

$$(-\tfrac{1}{2} V_{ij}{}^{kl} - \tfrac{1}{4} i e^{kl}{}_{mn} V_{ij}{}^{mn}) = \tfrac{1}{2}(U_i{}^k U_j{}^l - U_i{}^l U_j{}^k) \tag{5.20}$$

Equations (5.20) and (5.17) now serve to define $V_{ij}{}^{kl}$, and thus $V_{\Delta\Sigma}$, in terms of the fundamental W matrices.

A tedious insertion of the definitions (3.14) and (3.19) into the quantum condition (5.12) produces

$$\begin{aligned} -16\pi^2 n^2 &= -\frac{1}{2}\int_V R^*_{Pij}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta}\, dV_\alpha \\ &\quad - \frac{ie_{ij}{}^{kl}}{4} \int_V R^*_{Pkl}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta}\, dV_\alpha \\ &= -\frac{1}{2}\int_V R^*_{Pij}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta}\, dV_\alpha - \frac{i}{2}\int_V {}^*R^*_{Pij}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta}\, dV_\alpha \end{aligned} \tag{5.21}$$

where the second form is by the definition of the double dual Riemann tensor. Since the local affinity has previously been taken as real (certainly a realistic assumption), (3.17) shows that we must take as real the curvature tensor also, forcing the single equation (5.21) to become the two conditions

$$\int_V R^*_{Pij}{}^{\sigma\tau}{}_{:\tau} dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta} dV_\alpha = 32\pi^2 n^2 \tag{5.22a}$$

$$\int_V {}^*R^*_{Pij}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma \int_V R^{*ij\alpha\beta}_P{}_{:\beta}\, dV_\alpha = 0 \tag{5.22b}$$

To summarize, the consistency of the path-dependent treatment of the interactions of fields with gravity dictates that the two conditions (5.22) must everywhere apply; this is in contrast to the electromagnetic case, where only one such restriction on the theory holds.

For future reference we point out that equations (5.22) are completely free of dependence on or even existence of a tetrad $h_i{}^\mu$ defined over space-time. This is so because

$$R\overset{*}{P}_{ij}{}^{\sigma\tau}{}_{:\tau}\, dV_\sigma = (\tfrac{1}{2}\epsilon^{\sigma\tau\alpha\beta}R_{Pij\alpha\beta,\tau})(\epsilon_{\sigma\gamma\delta\epsilon}\, dx^\gamma\, dy^\delta\, dz^\epsilon)$$

Since

$$\epsilon^{\sigma\tau\alpha\beta} = he^{\sigma\tau\alpha\beta}$$

and

$$\epsilon_{\sigma\gamma\delta\epsilon} = (1/h)\, e_{\sigma\gamma\delta\epsilon}$$

the dependence on the determinant of $h_i{}^\mu$ cancels in the above expression. Moreover, none of the other quantities appearing are inherently tetrad dependent. Therefore, the quantization conditions (5.22) are valid without regard to the existence of a global metric.

6. *The Path-Dependent Matter Field Equations*

In order that the quantum condition (5.22) not automatically be restricted to the trivial $n = 0$ case, it is necessary that we be able to consistently ignore the original definition (3.17) of $R_{ij\mu\nu}$ in terms of $A_{ij\mu}$; in other words, if (3.17) defines the curvature tensor, the Bianchi identity inevitably results.[1] Consequently, we are forced to take the Riemann tensor as the fundamental dynamical variable of the gravitational field, rather than the potentials $A_{ij\mu}$. This idea is of course central to the question that motivated Mandelstam's original development of the path-dependent approach to gravitation (1962b); it suggests that the entity to be quantized in any quantum field theory of gravitation is the curvature tensor rather than the metric or affine connection. Equations (4.19) and (4.20) represent a good start towards a consistent affinity-free treatment of gravitation. It remains, however, to verify a path-dependent relation equivalent to (3.18).

To this end, following Mandelstam (1962a), let A, B, C, and D stand for the four space-time points x^μ, $x^\mu + dx^\mu$, $x^\mu + dy^\mu$, and $x^\mu + dx^\mu + dy^\mu$, respectively. Then,

$$\Psi(x, P)_{,\mu} = \frac{\Psi(x + dx, P + \mathrm{AB}) - \Psi(x, P)}{dx^\mu}$$

[1] Actually, this is not strictly true if we allow multiply connected space-times; for novel topologies such as "wormholes" it is possible to have the global quantity consisting of the integral of the Riemann tensor over a closed surface nonvanishing even though the Bianchi identity is locally satisfied everywhere. This is conceivable because in such space-times a closed surface may not enclose a well-defined interior region, making an application of Gauss' theorem impossible. We shall not consider this possibility here; for a general discussion, see Lubkin (1963).

where AB means the path segment from A to B. Thus,

$$\Psi(x,P)_{,\mu,\nu} = [\Psi(x+dx,P+\text{AB})_{,\nu} - \Psi(x,P)_{,\nu}]/dx^{\mu}$$

$$= \left\{ \left[\frac{\Psi(x+dx+dy,P+\text{ABD}) - \Psi(x+dx,P+\text{AB})}{dy^{\nu}} \right] - \left[\frac{\Psi(x+dy,P+\text{AC}) - \Psi(x,P)}{dy^{\nu}} \right] \right\} \Big/ dx^{\mu}$$

$$= \frac{\Psi(x+dx+dy,P+\text{ABD}) - \Psi(x+dx,P+\text{AB}) - \Psi(x+dy,P+\text{AC}) + \Psi(x,P)}{dx^{\mu}\,dy^{\nu}} \tag{6.1}$$

Here, ABD is the path segment from A to D through B. Similarly, we find

$$\Psi(x,P)_{,\nu,\mu} = \frac{\Psi(x+dy+dx,P+\text{ACD}) - \Psi(x+dy,P+\text{AC}) - \Psi(x+dx,P+\text{AB}) + \Psi(x,P)}{dy^{\nu}\,dx^{\mu}} \tag{6.2}$$

Subtracting (6.2) from (6.1), we obtain

$$\Psi(x,P)_{,\mu,\nu} - \Psi(x,P)_{,\nu,\mu} = \frac{\Psi(x+dx+dy,P+\text{ABD}) - \Psi(x+dx+dy,P+\text{ACD})}{dx^{\mu}\,dy^{\nu}} \tag{6.3}$$

Indicating by $dS^{\alpha\beta}$ the infinitesimal area bounded by the closed path ABDCA, we may use (4.21) to reexpress the numerator on the right of (6.3), yielding

$$\Psi(x,P)_{,\mu,\nu} - \Psi(x,P)_{,\nu,\mu} = \frac{\frac{1}{4}\boldsymbol{\tau}\cdot\mathbf{R}_{P\alpha\beta}(x+dx+dy)\,dS^{\alpha\beta}\Psi(x+dx+dy,P+\text{ACD})}{dx^{\mu}\,dy^{\nu}}$$

$$= \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu}(x)\Psi(x,P) \tag{6.4}$$

in the limit $dx, dy \to 0$, since

$$\mathbf{R}_{P\alpha\beta}\,dS^{\alpha\beta} = \mathbf{R}_{P\alpha\beta}(dx^{\alpha}\,dy^{\beta} - dx^{\beta}\,dy^{\alpha}) = 2\mathbf{R}_{P\alpha\beta}\,dx^{\alpha}\,dy^{\beta}$$

Note that we have derived the commutation relation (6.4) for derivatives of path-dependent fields directly from the properties of those fields, rather than through the formality of converting (3.18) to a path dependent equation by use of (4.8).

In path-dependent electromagnetism, the potentials A_μ are often useful auxillary variables, but are nonetheless completely irrelevant to the formulation of the field equations. In contrast, whereas the local affinity $A_{ij\mu}$ may likewise be completely dispensed with in path-dependent gravitation, the tetrad $h^i{}_\mu$ is essential since only through it are we able to determine the locally inertial frames at each point in space-time. How then are we to define the tetrad field in terms of the curvature tensor, as compared to the normal procedure of defining the curvature tensor as a certain combination of derivatives of the tetrad?

To answer this question, let us define the path-dependent tetrad $h_P{}^i{}_\mu$ in obvious fashion by

$$h_P{}^i{}_\mu(x) \equiv U^i{}_j(x, P) h^j{}_\mu(x) \tag{6.5}$$

Then, by definition of $U^i{}_j$,

$$h_P{}^i{}_{\mu,\nu} = U^i{}_j(h^j{}_{\mu|\nu}) \equiv U^i{}_j(h^j{}_{\mu,\nu} + A^j{}_{k\nu} h^k{}_\mu)$$

According to (2.5), this is equivalent to

$$\begin{aligned} h_P{}^i{}_{\mu,\nu} &= U^i{}_j h^j{}_\lambda \Gamma^\lambda{}_{\mu\nu} \\ &= h_P{}^i{}_\lambda \Gamma^\lambda{}_{\mu\nu} \end{aligned} \tag{6.6}$$

Note the striking similarity between (6.6) and the relation (5.18) satisfied by $U^i{}_j$. This similarity points up the intimate connection between the tetrad and path-dependent formulations of gravitation: $h_P{}^i{}_\mu$ may be viewed alternatively as the transformation coefficients which convert global tensors to path-dependent, generalized Lorentz invariant local quantities, or as the path-dependent matrix that changes ordinary global tensors to global quantities that are unchanging under the *general* coordinate transformation group $GL(4, R)$.

Like Ψ, $h_P{}^i{}_\mu$ is a path-dependent, Lorentz invariant quantity, so it too must obey relations similar to (4.20) and (6.4). In particular,

$$h_P{}^i{}_{\mu,\nu,\sigma} - h_P{}^i{}_{\mu,\sigma,\nu} = h_P{}^j{}_\mu R_P{}^i{}_{j\nu\sigma} \tag{6.7}$$

or

$$h_P{}^{j\mu}(h_P{}^i{}_{\mu,\nu,\sigma} - h_P{}^i{}_{\mu,\sigma,\nu}) = R_P{}^{ij}{}_{\nu\sigma} \tag{6.8}$$

Unlike Ψ, however, we do not postulate an additional equation of motion for the tetrad to satisfy; instead, we take (6.8) as the "field equation" for $h_P{}^i{}_\mu$. Because our point of view is that $R_P{}^{ij}{}_{\nu\sigma}$ is the fundamental gravitational field variable, (6.8) should be viewed as a definition of $h_P{}^i{}_\mu$ in terms of $R_P{}^{ij}{}_{\nu\sigma}$ rather than vice-versa. Since (6.8) is a second-order differential equation, we require two boundary conditions in order to uniquely specify the tetrad; these are provided by (2.3). In addition, since we are neglecting torsion, we must demand the symmetry of the global affine connection:

$$h_P{}^i{}_{\mu,\nu} = h_P{}^i{}_{\nu,\mu} \tag{6.9}$$

We are at last in a position to state all of the general path dependence equations, save those for the curvature tensor itself. Therefore, we ignore the origins of the theory so far developed and instead concentrate full attention on the path-dependent matter fields $\Psi(x, P)$. Summarizing,[2] we have

$$\Psi^{A\cdots\dot{A}\cdots}(x, P') = M^A{}_B(\Sigma)\cdots M^{\dot{A}}{}_{\dot{B}}(\Sigma)\cdots\Psi^{B\cdots\dot{B}\cdots}(x, P) \tag{6.10a}$$

$$M^A{}_B(\Sigma) = \delta^A_B + \frac{1}{4}\int_\Sigma M^A{}_C(\xi, x)\boldsymbol{\tau}^C{}_B\cdot\mathbf{R}_{P\mu\nu}(\xi)\,dS^{\mu\nu}(\xi) \tag{6.10b}$$

$$M^{\dot{A}}{}_{\dot{B}}(\Sigma) = \delta^{\dot{A}}_{\dot{B}} + \frac{1}{4}\int_\Sigma M^{\dot{A}}{}_{\dot{C}}(\xi, x)\boldsymbol{\tau}^{\dot{C}}{}_{\dot{B}}\cdot\overline{\mathbf{R}_{P\mu\nu}}(\xi)\,dS^{\mu\nu}(\xi) \tag{6.10c}$$

$$\Psi^{\dot{A}\cdots A\cdots}{}_{,\mu,\nu} - \Psi^{A\cdots\dot{A}\cdots}{}_{,\nu,\mu} = \tfrac{1}{2}(\boldsymbol{\tau}^A{}_B\cdot\mathbf{R}_{P\mu\nu} + \cdots + \boldsymbol{\tau}^{\dot{A}}{}_{\dot{B}}\cdot\overline{\mathbf{R}_{P\mu\nu}})\Psi^{B\cdots\dot{B}\cdots} \tag{6.10d}$$

$$h_P{}^{j\mu}(h_P{}^i{}_{\mu,\nu,\sigma} - h_P{}^i{}_{\mu,\sigma,\nu}) = R_P{}^{ij}{}_{\nu\sigma} \tag{6.10e}$$

$$h_P{}^i{}_{\mu,\nu} = h_P{}^i{}_{\nu,\mu} \tag{6.10f}$$

$$h_P{}^i{}_\mu(0) = \delta^i{}_\mu \tag{6.10g}$$

$$h_P{}^i{}_{\mu,\sigma}(0) = 0 \tag{6.10h}$$

$$h_P{}^i{}_\mu h_P{}^{j\mu} = \eta^{ij} \tag{6.10i}$$

Here Σ is any surface bounded by the closed path $P' - P$.

We see equations (6.10) obviate entirely the need for the local affinity $A_{ij\mu}$. Observe also that (6.10a), (6.10b), and (6.10c) lead to a unique dependence of Ψ on the gravitational field only as a result of the consistency condition imposed in Section 5.

To the list (6.10) we need add propagation equations for Ψ. These are provided by the well-known Dirac–Fierz–Pauli wave equations (see, e.g., Roman, 1960) of flat space-time, generalized through the prescription that all matrices $\sigma^i{}_{A\dot{B}}$ be replaced by $h_{Pi}{}^\mu\sigma^i{}_{A\dot{B}}$. We emphasize that because Ψ contains path dependency, ordinary flat-space derivatives, rather than becoming covariant derivatives, *remain* ordinary derivatives when rewriting the field equations for Ψ in curved space-time. [However, according to (6.10d) these ordinary differentiations do not commute when space-time is not flat.]

As an example, a typical DFP wave equation in flat space-time takes the form

$$\sigma^{iE}{}_{\dot{F}}\,\partial_i\psi^{\dot{F}} + (im/\sqrt{2}\hbar)\phi^E = 0$$

Applying the given rule, this generalizes in curved space-time to

$$h_{Pi}{}^\mu\sigma^{iE}{}_{\dot{F}}\,\partial_\mu\Psi^{\dot{F}} + (im/\sqrt{2}\hbar)\Phi^E = 0 \tag{6.11}$$

[2] Equation (6.10i) breaks down at those isolated points where $\mathbf{R}^{*\mu\nu}_P{}_{:\nu} \neq 0$; this circumstance will be considered in the next section.

Now, according to (5.17) and (6.5)

$$
\begin{aligned}
h_{Pi}{}^{\mu}\sigma^{iE}{}_{\dot F} &= h_j{}^{\mu}\sigma_{iA}{}^{\dot B}W^A{}_C\sigma^{jC}{}_{\dot D}W^{-1\dot D}{}_{\dot B}\sigma^{iE}{}_{\dot F}\\
&= h_j{}^{\mu}\sigma_{iA}{}^{\dot B}\sigma^{iE}{}_{\dot F}W^A{}_C\sigma^{jC}{}_{\dot D}W^{-1\dot D}{}_{\dot B}\\
&= h_j{}^{\mu}\delta^E_A\delta^{\dot B}_{\dot F}W^A{}_C\sigma^{jC}{}_{\dot D}W^{-1\dot D}{}_{\dot B}
\end{aligned}
$$

since (Bade & Jehle, 1953)

$$\sigma_{iA}{}^{\dot B}\sigma^{iE}{}_{\dot F} = \delta^E_A\delta^{\dot B}_{\dot F}$$

Thus, we have

$$h_{Pi}{}^{\mu}\sigma^{iE}{}_{\dot F} = h_j{}^{\mu}W^E{}_C\sigma^{jC}{}_{\dot D}W^{-1\dot D}{}_{\dot F}$$

In addition, we have

$$\Phi^E = W^E{}_C\phi^C$$

$$\partial_\mu\Psi^{\dot F} = \partial_\mu(W^{\dot F}{}_{\dot H}\psi^{\dot H}) = W^{\dot F}{}_{\dot H}(\psi^{\dot H}{}_{|\mu})$$

Substituting these results into (6.11), we obtain

$$
\begin{aligned}
0 &= h_j{}^{\mu}W^E{}_C\sigma^{jC}{}_{\dot D}W^{-1\dot D}{}_{\dot F}W^{\dot F}{}_{\dot H}(\psi^{\dot H}{}_{|\mu}) + (im/\sqrt{2}\hbar)W^E{}_C\phi^C\\
&= h_j{}^{\mu}W^E{}_C\sigma^{jC}{}_{\dot D}\delta^{\dot D}_{\dot H}\psi^{\dot H}{}_{|\mu} + (im/\sqrt{2}\hbar)W^E{}_C\phi^C\\
&= h_j{}^{\mu}W^E{}_C\sigma^{jC}{}_{\dot D}\psi^{\dot D}{}_{|\mu} + (im/\sqrt{2}\hbar)W^E{}_C\phi^C\\
&= W^E{}_C(h_j{}^{\mu}\sigma^{jC}{}_{\dot D}\psi^{\dot D}{}_{|\mu} + (im/\sqrt{2}\hbar)\phi^C)\\
&= h_j{}^{\mu}\sigma^{jC}{}_{\dot D}\psi^{\dot D}{}_{|\mu} + (im/\sqrt{2}\hbar)\phi^C
\end{aligned}
\tag{6.12}
$$

Since (6.12) is just the generally covariant DFP equation of motion, we have substantiated the assertion that if the path-dependent field Ψ obeys (6.11), the associated ordinary field ψ is a solution of the generally covariant curved space DFP equation.

7. *The Dirac Veto*

Equation (6.4) leads to a further restriction on the path-dependent fields, as will now be demonstrated. Differentiating (6.4), we obtain

$$\Psi_{,\mu,\nu,\sigma} - \Psi_{,\nu,\mu,\sigma} = \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu,\sigma}\Psi + \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu}\Psi_{,\sigma} \tag{7.1}$$

Cyclically permuting $\mu\nu\sigma$ in (7.1) produces the two further equations

$$\Psi_{,\sigma,\mu,\nu} - \Psi_{,\mu,\sigma,\nu} = \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\sigma\mu,\nu}\Psi + \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\sigma\mu}\Psi_{,\nu} \tag{7.2}$$

$$\Psi_{,\nu,\sigma,\mu} - \Psi_{,\sigma,\nu,\mu} = \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\nu\sigma,\mu}\Psi + \tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{R}_{P\nu\sigma}\Psi_{,\mu} \tag{7.3}$$

Adding (7.1), (7.2), and (7.3), and grouping terms, we obtain

$$(\Psi_{,\mu,\nu,\sigma} - \Psi_{,\mu,\sigma,\nu}) + (\Psi_{,\sigma,\mu,\nu} - \Psi_{,\sigma,\nu,\mu})$$
$$+ (\Psi_{,\nu,\sigma,\mu} - \Psi_{,\nu,\mu,\sigma}) = \tfrac{1}{2}\boldsymbol{\tau}\cdot(\mathbf{R}_{P\mu\nu,\sigma} + \mathbf{R}_{P\sigma\mu,\nu} + \mathbf{R}_{P\nu\sigma,\mu})\Psi$$
$$+ \tfrac{1}{2}\boldsymbol{\tau}\cdot(\mathbf{R}_{P\mu\nu}\Psi_{,\sigma} + \mathbf{R}_{P\sigma\mu}\Psi_{,\nu} + \mathbf{R}_{P\nu\sigma}\Psi_{,\mu})$$

The terms in parentheses on the left in the preceding may be rewritten using (6.4); we find

$$\tfrac{1}{2}\boldsymbol{\tau}\cdot(\mathbf{R}_{P\nu\sigma}\Psi_{,\mu} + \mathbf{R}_{P\mu\nu}\Psi_{,\sigma} + \mathbf{R}_{P\sigma\mu}\Psi_{,\nu}) = \tfrac{1}{2}\boldsymbol{\tau}\cdot(\mathbf{R}_{P\mu\nu,\sigma} + \mathbf{R}_{P\sigma\mu,\nu}$$
$$+ \mathbf{R}_{P\nu\sigma,\mu})\Psi + \tfrac{1}{2}\boldsymbol{\tau}\cdot(\mathbf{R}_{P\mu\nu}\Psi_{,\sigma} + \mathbf{R}_{P\sigma\mu}\Psi_{,\nu} + \mathbf{R}_{P\nu\sigma}\Psi_{,\mu})$$

or

$$\boldsymbol{\tau}\cdot(\mathbf{R}_{P\mu\nu,\sigma} + \mathbf{R}_{P\sigma\mu,\nu} + \mathbf{R}_{P\nu\sigma,\mu})\Psi = 0$$

Contracting this with the Levi-Civita tensor density $e^{\alpha\mu\nu\sigma}$ (the reason for not using the tensor $\epsilon^{\alpha\mu\nu\sigma}$ will become evident),

$$e^{\alpha\mu\nu\sigma}\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu,\sigma}\Psi = 0 \tag{7.4}$$

If y is one of those isolated points in space where $e^{\alpha\mu\nu\sigma}\mathbf{R}_{P\mu\nu,\sigma} \neq 0$ as discussed in Section 5, (7.4) requires that $\Psi(y) = 0$. More generally, we conclude that Ψ *must vanish along the worldline of a source of the dual Riemann tensor.*

Of course, this sort of restriction occurs in *any* gauge field theory generalized to include "dual" charges. In the particular case of electromagnetism, we find that the requirement takes the form that the wave functions of electrically charged matter must vanish wherever the fields representing magnetically charged matter are nonzero. Wentzel (1966) has aptly dubbed this restriction the *Dirac veto.* If we denote by D the set of all space-time points lying on worldlines of sources of the dual Riemann tensor, the Dirac veto in gravitation may be summarized symbolically as

$$\Psi(D) = 0 \tag{7.5}$$

However, the Dirac veto leads to peculiarities in gravitation not exhibited in monopole electrodynamics. Since equation (6.7) for the commutation of derivatives of the tetrad is exactly analogous to (5.4) from which flows (7.5), it follows by identical logic that $h_P{}^i{}_\mu$ must itself be subject to the Dirac veto. Specifically, from (6.7) we may obtain an equation similar to (7.4); namely,

$$e^{\alpha\nu\sigma\tau}R_P{}^i{}_{j\nu\sigma,\tau}h_P{}^j{}_\mu = 0 \tag{7.6}$$

from which it follows

$$h_P{}^i{}_\mu(D) = 0 \tag{7.7}$$

Arguing in the same fashion for $h_P{}^{j\nu}$, we find

$$h_P{}^{j\nu}(D) = 0 \tag{7.8}$$

Taken together, (7.7) and (7.8) imply the breakdown on D of definition (2.1b). Obviously, our normal notion of space-time must fail at points in D. This is not, however, particularly disturbing, for two reasons.

First, because all matter fields must vanish on D according to the Dirac veto, the points in D cannot be directly probed experimentally. Second, although (7.7) and (7.8) must certainly hold, we have not required that $h_P{}^i{}_\mu$ be *continuous* on D. Indeed, since the behavior of the divergence of the dual Riemann tensor is highly discontinuous at points in D, we have no reason to expect or demand that the tetrad be continuous at those same points. Thus, the tetrad may behave reasonably in any neighborhood of a dual source worldline, and yet vanish on the worldline.

Since $h_{Pa}{}^\alpha(D) = 0$,

$$h(D) \equiv \det[h_{Pa}{}^\alpha(D)] = 0$$

Because the totally contravariant Levi-Civita tensor is proportional to h, it also vanishes on D. On the other hand, the constant numerical array $e^{\alpha\beta\gamma\delta}$ is nonzero even on D; this is the reason for employing the tensor density $e^{\alpha\beta\gamma\delta}$ rather than the tensor $\epsilon^{\alpha\beta\gamma\delta}$ in formulating conditions (7.4) and (7.6). Moreover, in view of (7.7) and (7.8), it is now possible to appreciate the tetrad (and thus global metric) independence of the quantum conditions, as brought out at the end of section 5. Because of this independence, conditions (5.22) are meaningful even on D.

Because non-Abelian gauge fields self-couple, the path-dependent curvature tensor itself obeys the appropriate form of (6.4), namely,

$$R_{Pij\alpha\beta,\mu,\nu} - R_{Pij\alpha\beta,\nu,\mu} = -R_P{}^k{}_{i\mu\nu}R_{Pkj\alpha\beta} - R_P{}^k{}_{j\mu\nu}R_{Pik\alpha\beta}$$

From this, by reasoning parallel to that leading to (7.4), we arrive at

$$e^{\tau\mu\nu\sigma}R_P{}^k{}_{i\mu\nu,\sigma}R_{Pkj\alpha\beta} + e^{\tau\mu\nu\sigma}R_P{}^k{}_{j\mu\nu,\sigma}R_{Pik\alpha\beta} = 0 \tag{7.9}$$

which is evidently satisfied if

$$R_{Pij\alpha\beta}(D) = 0 \tag{7.10}$$

The curvature tensor is therefore also subject to the Dirac veto.

Intuitively, we would expect the field strength to assume arbitrarily large values in the vicinity of a point monopole. Condition (7.10) does not contradict this, since (as with the tetrad) it is impossible to measure the field strength at the point occupied by the dual source and there is no reason to expect continuity of the field there.

As a final application of the Dirac veto, we may begin with

$$R_{Pij\alpha\beta,\gamma,\mu,\nu} - R_{Pij\alpha\beta,\gamma,\nu,\mu} = -R_P{}^k{}_{i\mu\nu}R_{Pkj\alpha\beta,\gamma} - R_P{}^k{}_{j\mu\nu}R_{Pik\alpha\beta,\gamma}$$

and obtain

$$e^{\tau\mu\nu\sigma}R_P{}^k{}_{i\mu\nu,\sigma}R_{Pkj\alpha\beta,\gamma} + e^{\tau\mu\nu\sigma}R_P{}^k{}_{j\mu\nu,\sigma}R_{Pik\alpha\beta,\gamma} = 0 \tag{7.11}$$

[This may also be easily seen by differentiating (7.9) and imposing (7.10).]

We could satisfy (7.11) by taking

$$R_{Pij\alpha\beta,\gamma}(D) = 0$$

but this would in turn lead to

$$e^{\tau\alpha\beta\gamma}R_{Pij\alpha\beta,\gamma}(D) = 0$$

which would obviously render the whole analysis to this point an empty exercise.

This dilemma is easily resolved: We simply ignore (7.11). We can give some justification to this seeming inconsistency. Since $e^{\tau\mu\nu\sigma}R_{Pij\mu\nu,\sigma}$ is nonvanishing only on D, (7.11) is nontrivial only at those isolated points in space which serve as sources of the dual Riemann tensor. At these points, according to the developments of section 5, $R_{Pij\mu\nu,\sigma}$ must have the behavior of a three-space δ functional. On D, therefore, the left-hand side of (7.11) corresponds to a product of two δ functionals; the product of two such distributions is well known to be meaningless mathematically. If nothing else, we may conclude that it is imprudent to inquire too closely as to the Dirac veto status of the derivative of the curvature tensor!

8. *The Path-Dependent Gravitational Field Equations*

As yet we have not discussed the field equations which pertain specifically to gravitation. Clearly, the ordinary equivalent forms of the Einstein equation

$$*R*^{\alpha}{}_{\mu\alpha\nu} = G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu} \tag{8.1a}$$

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \tag{8.1b}$$

where

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}, \qquad R \equiv R^{\mu}{}_{\mu}, \qquad T \equiv T^{\mu}{}_{\mu}$$

are unsuitable. This is evident because our goal is to consider the curvature tensor as the premier field describing gravity and not, as is the custom, to assume that $R_{ij\mu\nu}$ is derivable from potentials. Hence, we require an equation involving *derivatives* of the Riemann tensor in order that, given an initial configuration, the field may be propagated throughout space-time; obviously (8.1) does not meet this criterion. Moreover, (8.1) provides no hint as to the generalization of the Bianchi identity to include a source term for the dual curvature tensor. Consequently, we must seek new field equations for gravitation. However, in order to retain the successes of general relativity, we shall continue to accept (8.1) as an *algebraic* relation satisfied by the Riemann tensor.

Once again, we fall back on the example of our model field, electromagnetism. The normal Maxwell equations

$$F^{\mu\nu}{}_{,\nu} = j^{\mu} \tag{8.2a}$$

$$F^{*\mu\nu}{}_{,\nu} = 0 \tag{8.2b}$$

generalize in the presence of a magnetic current k^μ to

$$F^{\mu\nu}{}_{,\nu} = j^\mu \tag{8.3a}$$

$$F^{*\mu\nu}{}_{,\nu} = k^\mu \tag{8.3b}$$

In order to find analogous equations for the curvature tensor, we consider the identity

$$R_{\mu\sigma;\tau} - R_{\mu\nu;\sigma} - R_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = G_\sigma{}^\lambda{}_{;\lambda}g_{\tau\mu} - G_\tau{}^\lambda{}_{;\lambda}g_{\sigma\mu} + {}^*R^*{}_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} \tag{8.4}$$

which is obtained after a bit of work from the definition of the double dual Riemann tensor and the properties of the Levi–Civita tensor. In the broader context of non-Abelian gauge fields, this type of identity is peculiar to gravitation since only in this case is it possible to convert the Yang–Mills indices ij in the field tensor to space-time indices $\mu\nu$.

The Bianchi identity

$${}^*R^*{}_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = 0 = G_\sigma{}^\lambda{}_{;\lambda}$$

holds in normal Riemann space-times, in which case (8.4) reduces to the two equations

$$R_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = R_{\mu\sigma;\tau} - R_{\mu\tau;\sigma} \tag{8.5a}$$

$${}^*R^*{}_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = 0 \tag{8.5b}$$

Since the right-hand side of (8.5a) may be written in terms of the source tensor $T_{\alpha\beta}$ by virtue of (8.1b), equations (8.5) bear a strong resemblance to the Maxwell equations (8.2) and are often referred to as the *quasi-Maxwell equations of gravitation.* Indeed, within the framework of normal general relativity, Lichnerowicz (1960) has proved that if the Riemann tensor satisfies (8.1) on an initial spacelike hypersurface S, then the solutions of (8.5) satisfy the Einstein equation (8.1) throughout some neighborhood of S. Hence, from our point of view the equations (8.5) are the perfect candidates for the field equations of gravity in a space-time for which the Bianchi identity is valid.

At this point our strict analogy with the electromagnetic field breaks down. If we postulate a source term $K_{\sigma\tau\mu}$ for the double dual Riemann tensor as suggested by reference to the generalized Maxwell equations (8.3), the generalized quasi-Maxwell equations are forced as a consequence of the identity (8.4) (of which there is no counterpart in electrodynamics) to take the unsymmetrical form

$$R_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = J_{\mu\sigma\tau} - K_{\mu\sigma\tau} \tag{8.6a}$$

$${}^*R^*{}_{\sigma\tau\mu}{}^\lambda{}_{;\lambda} = K_{\mu\sigma\tau} \tag{8.6b}$$

where

$$J_{\mu\sigma\tau} \equiv R_{\mu\sigma;\tau} - R_{\mu\tau;\sigma}$$

It is apparently not possible in gravitation to maintain the symmetry of the field equations in the same sense as the symmetry exhibited in the generalized Maxwell equations between the electric and "dual" (magnetic) currents.

Moreover, equations (8.6) lead to fundamental difficulties with the geometric interpretation of gravity. If we posit "dual" matter as the source of the dual Riemann tensor in accordance with (8.6b), then by analogy to the two-potential approach to the generalized Maxwell equations (Cabibbo & Ferrari, 1962), we could introduce two affine connections Γ and $\tilde{\Gamma}$ (and two metrics g and $\tilde{g}$) such that the Riemann tensor is given by

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\mu,\nu} - \Gamma^{\alpha}{}_{\beta\nu,\mu} + \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu} - \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu} - \epsilon_{\mu\nu}{}^{\sigma\tau}(\tilde{\Gamma}^{\alpha}{}_{\beta\sigma,\tau} + \tilde{\Gamma}^{\alpha}{}_{\lambda\tau}\tilde{\Gamma}^{\lambda}{}_{\beta\sigma})$$

$$R^{*\alpha}{}_{\beta\mu\nu} = \tilde{\Gamma}^{\alpha}{}_{\beta\mu,\nu} - \tilde{\Gamma}^{\alpha}{}_{\beta\nu,\mu} + \tilde{\Gamma}^{\alpha}{}_{\lambda\nu}\tilde{\Gamma}^{\lambda}{}_{\beta\mu} - \tilde{\Gamma}^{\alpha}{}_{\lambda\mu}\tilde{\Gamma}^{\lambda}{}_{\beta\nu} + \epsilon_{\mu\nu}{}^{\sigma\tau}(\Gamma^{\alpha}{}_{\beta\sigma,\tau} + \Gamma^{\alpha}{}_{\lambda\tau}\Gamma^{\lambda}{}_{\beta\sigma})$$

Ordinary matter would move along geodesics given by Γ and "dual" matter along a different set of geodesics described by $\tilde{\Gamma}$. We would therefore have the strange situation of two distinct types of matter interacting with two coexisting, superimposed geometries, both of which are ultimately described by one field, the curvature tensor. Further, with two metrics we would have an embarrassing plethora of tensors from which to choose the physically significant ones. From every rank-two tensor, two distinct contractions could be formed; every contravariant vector would give rise to two covariant vectors.

Rather than pursue these lines of conjecture, we reject equations (8.6). Instead, we attempt to decouple the identity (8.4) to yield equations similar to, but more general than, (8.5), and without the introduction of the *ad hoc* quantities appearing in (8.6).Examination of (8.4) shows that this may be accomplished uniquely:

$$R_{\sigma\tau\mu}{}^{\lambda}{}_{;\lambda} = R_{\mu\sigma;\tau} - R_{\mu\tau;\sigma} + \tfrac{2}{3}(G_{\tau}{}^{\lambda}{}_{;\lambda}g_{\sigma\mu} - G_{\sigma}{}^{\lambda}{}_{;\lambda}g_{\tau\mu}) \tag{8.7a}$$

$$^{*}R^{*}{}_{\sigma\tau\mu}{}^{\lambda}{}_{;\lambda} = \tfrac{1}{3}(G_{\tau}{}^{\lambda}{}_{;\lambda}g_{\sigma\mu} - G_{\sigma}{}^{\lambda}{}_{;\lambda}g_{\tau\mu}) \tag{8.7b}$$

The coefficients $\frac{2}{3}$ and $\frac{1}{3}$ are established by contracting the equations over σ and μ. Equations (8.7) represent the simplest consistent generalization of equations (8.5) and will be taken as the generalized gravitational field equations.

By (8.1a), $G_{\mu\nu}$ is proportional to the energy-momentum tensor $T_{\mu\nu}$. As a consequence, the right-hand side of (8.7b) is nonvanishing if and only if $T_{\mu}{}^{\nu}{}_{;\nu} \neq 0$; that is, *the sources of the dual Riemann tensor are violations of local energy-momentum conservation.* We are accustomed to saying that the vanishing of the divergence of the dual Riemann tensor leads to energy conservation; (8.7b) compels the stronger conclusion that the divergence of the dual Riemann tensor is zero *if and only if* energy is locally conserved. As shown in the next section, (8.7b) possesses the additional desirable property of satisfying identically the quantization requirement (5.22b).

The vanishing of the general covariant derivative of the tetrad according to (2.5) implies that conversion of global to local indices commutes with general covariant differentiation; (8.7) may thus be written as

$$R_{ij}{}^{\mu\lambda}{}_{;\lambda} = R^{\mu}{}_{i;j} - R^{\mu}{}_{j;i} + \tfrac{2}{3}(G_{j}{}^{\lambda}{}_{;\lambda}h_{i}{}^{\mu} - G_{i}{}^{\lambda}{}_{;\lambda}h_{j}{}^{\mu})$$

$$^{*}R^{*}{}_{ij}{}^{\mu\lambda}{}_{;\lambda} = \tfrac{1}{3}(G_{j}{}^{\lambda}{}_{;\lambda}h_{i}{}^{\mu} - G_{i}{}^{\lambda}{}_{;\lambda}h_{j}{}^{\mu})$$

Converting to path-dependent quantities and global covariant derivatives, and in addition inserting the source terms by (8.1), we arrive at the path-dependent field equations

$$R_{Pij}{}^{\mu\lambda}{}_{:\lambda} = -8\pi G[(T_P{}^{\mu}{}_{i:j} - T_P{}^{\mu}{}_{j:i}) - \tfrac{1}{2}(h_{Pi}{}^{\mu}T_{P,j} - h_{Pj}{}^{\mu}T_{P,i}) + \tfrac{2}{3}(T_{Pj}{}^{\lambda}{}_{:\lambda}h_{Pi}{}^{\mu} - T_{Pi}{}^{\lambda}{}_{:\lambda}h_{Pj}{}^{\mu})] \tag{8.8a}$$

$$*R\sharp_{ij}{}^{\mu\lambda}{}_{:\lambda} = -(8\pi G/3)(T_{Pj}{}^{\lambda}{}_{:\lambda}h_{Pi}{}^{\mu} - T_{Pi}{}^{\lambda}{}_{:\lambda}h_{Pj}{}^{\mu}) \tag{8.8b}$$

(Note that while we use the notation T_P for the energy density scalar, actually $T_P = T$ since scalars are by definition Lorentz invariant and thus path independent.)

Evaluated at points not in D (i.e., at most points in three-space), relations (8.8) reduce to the path-dependent quasi-Maxwell equations

$$R_{Pij}{}^{\mu\lambda}{}_{:\lambda} = -8\pi G[(T_P{}^{\mu}{}_{i:j} - T_P{}^{\mu}{}_{j:i}) - \tfrac{1}{2}(h_{Pi}{}^{\mu}T_{P,j} - h_{Pj}{}^{\mu}T_{P,i})]$$

$$*R\sharp_{ij}{}^{\mu\lambda}{}_{:\lambda} = 0$$

However, since both sides of (8.8a) and (8.8b) contain an essential dependence on the tetrad or its determinant, for points in D equations (8.8) collapse to the trivial identity $0 = 0$. Consequently, the differential equations (8.8) are not very useful on D as generalized quasi-Maxwell equations.

We can, however, integrate equations (8.8) over a three-volume V:

$$\int_V R_{Pij}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu = -8\pi G\left\{\int_V (T_P{}^{\mu}{}_{i:j} - T_P{}^{\mu}{}_{j:i})\, dV_\mu + \int_V \left[-\frac{1}{2}(T_{P,j}\, dV_i - T_{P,i}\, dV_j) + \frac{2}{3}(T_{Pj}{}^{\lambda}{}_{:\lambda}\, dV_i - T_{Pi}{}^{\lambda}{}_{:\lambda}\, dV_j)\right]\right\} \tag{8.9a}$$

$$\int_V *R\sharp_{ij}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu = \frac{-8\pi G}{3}\int_V (T_{Pj}{}^{\lambda}{}_{:\lambda}\, dV_i - T_{Pi}{}^{\lambda}{}_{:\lambda}\, dV_j) \tag{8.9b}$$

Equations (8.9) are valid even if V contains one or more dual source points; we take them as the final form of the generalized path-dependent gravitational field equations. Inasmuch as V may be a very small volume, (8.9) are basically equivalent to (8.8), but are free of the attendant difficulties caused by the Dirac veto. This situation is not unprecedented. A familiar example is the electric field equation for a point charge q

$$\nabla\cdot\mathbf{E} = 4\pi q\delta(\mathbf{x})$$

which is little more than a shorthand for

$$\int_V \nabla\cdot\mathbf{E}\, dV = 4\pi q$$

One further point warrants mention. If we take the divergence of the generalized Maxwell equations (8.3), we obtain the two charge conservation equations

$$j^{\mu}{}_{,\mu} = k^{\mu}{}_{,\mu} = 0$$

Similarly, by taking the divergence of either of the generalized quasi-Maxwell equations (8.7a) or (8.7b) and performing straightforward manipulations we arrive at the single "conservation" equation

$$G_{\alpha}{}^{\lambda}{}_{;\lambda;\beta} - G_{\beta}{}^{\lambda}{}_{;\lambda;\alpha} = 0$$

or, by (8.1a),

$$T_{\alpha}{}^{\lambda}{}_{;\lambda;\beta} - T_{\beta}{}^{\lambda}{}_{;\lambda;\alpha} = 0 \tag{8.10}$$

As with the generalized quasi-Maxwell equations (8.9), to ensure the non-triviality of (8.10) we must write this equation in integral form:

$$\int_V (T_{Pi}{}^{\lambda}{}_{:\lambda,j} - T_{Pj}{}^{\lambda}{}_{:\lambda,i})\, dV_{\mu} = 0 \tag{8.11}$$

Requirement (8.11) represents a restriction on the allowed energy-momentum tensor divergence; in equivalent terms, we may say that the four-vector of force density associated with energy nonconservation must be curl-free.

9. *Interpretation of the Quantum Condition*

The divergence of the energy-momentum tensor is the force density G_{Pi}, defined by

$$G_{Pi} \equiv T_{Pi}{}^{\lambda}{}_{:\lambda} = \partial p_{Pi}/\partial\Omega \tag{9.1}$$

where P_{Pi} is the energy-momentum four-vector and $d\Omega$ is the infinitesimal four-volume scalar given by

$$d\Omega = e_{ijkl}\, dw^i\, dx^j\, dy^k\, dz^l \tag{9.2}$$

The four vectors dw, dx, dy, and dz must be linearly independent in order that $d\Omega \neq 0$. Note that since the three-volume vector dV_i is given by

$$dV_i = e_{ijkl}\, dx^j\, dy^k\, dz^l \tag{9.3}$$

(9.2) is equivalent to

$$d\Omega = dw^i\, dV_i \tag{9.4}$$

where dw^i must not be contained in the three-space spanned by dx, dy, and dz.

Now

$$\int_V G_{Pi}\, dV_j = \int_V \frac{\partial p_{Pi}}{\partial\Omega}\, dV_j = \int_V \frac{\partial p_{Pi}}{\partial w^k\, \partial V_k}\, dV_j$$

by (9.1) and (9.4). Thus,

$$\int_V G_{Pi}\, dV_j = \int_V d\left(\frac{\partial p_{Pi}}{\partial w^k}\right)\delta^k_j = \frac{\partial p_{Pi}}{\partial w^j} \tag{9.5}$$

where dw^j must have a component in the direction of dV^i.

When performing three-space integrations it is most convenient to integrate over the hypersurface x^0 = const. Then, we may choose the three vectors dx, dy, and dz to lie along the x^1, x^2, and x^3 axes, respectively; in this case the only nonvanishing component of dV_i is

$$dV_0 = e_{0123}\, dx^1\, dx^2\, dx^3 = -dx^1\, dx^2\, dx^3 \equiv -d\tau \tag{9.6}$$

(since $e_{0123} = -1$). Since dV_i is therefore timelike, and because dw^j must have a component parallel to dV^l, we may choose

$$dw^j = dt \tag{9.7}$$

[Observe that for this choice of the hypersurface dV, (9.1) becomes

$$G_{Pi} = \frac{\partial p_{Pi}}{\partial \Omega} = \frac{-\partial p_{Pi}}{\partial t\, \partial \tau} = -\partial\left(\frac{\partial p_{Pi}}{\partial t}\right)\Big/ \partial\tau$$

which justifies the designation *force density* for G_{Pi}.]

In view of (9.7), (9.5) becomes

$$\int_V G_{Pi}\, dV_j = \begin{cases} \partial p_{Pi}/\partial t, & j = 0 \\ 0, & j = 1, 2, 3 \end{cases} \tag{9.8}$$

Thus, G_{Pi} integrated over a spacelike three-volume V corresponds to the time rate of change of four-momentum in V due to energy nonconservation.

Rewriting the gravitational field equation (8.9b) using (9.1), we obtain

$$\int_V {}^*R^*_{Pij}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu = \frac{-8\pi G}{3}\int_V (G_{Pj}\, dV_i - G_{Pi}\, dV_j) \tag{9.9}$$

by the properties of tensor duality

$$\int_V {}^{**}R^*_{Pab}{}^{\mu\lambda}{}_{;\lambda}\, dV_\mu \equiv \tfrac{1}{2} e_{ab}{}^{ij} \int_V {}^*R^*_{Pij}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu = -\int_V R^*_{Pab}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu \tag{9.10}$$

Combining (9.9) with (9.10), we obtain

$$-\int_V R^*_{Pab} = \frac{-8\pi G}{6} e_{ab}{}^{ij} \frac{-8\pi G}{6} e_{ab}{}^{ij} \int_V (G_{Pj}\, dV_i - G_{Pi}\, dV_j)$$

or

$$\int_V R^*_{Pab}{}^{\mu\lambda}{}_{:\lambda}\, dV_\mu = \frac{-8\pi G}{3} e_{ab}{}^{ij} \int_V G_{Pi}\, dV_j \tag{9.11}$$

By (9.9) and (9.11), we have

$$\int_V {}^*R^*_{Pij}{}^{\sigma\tau}{:}_\tau \, dV_\sigma \int_V R^{*ij\alpha\beta}_P{:}_\beta \, dV_\alpha$$
$$= \left(\frac{8\pi G}{3}\right)^2 \left[\int_V (G_{Pj}\, dV_i - G_{Pi}\, dV_j)\right] e^{ijkl} \int_V G_{Pk}\, dV_l$$
$$= -2\left(\frac{8\pi G}{3}\right)^2 e^{ijkl} \int_V G_{Pi}\, dV_j \int_V G_{Pk}\, dV_l$$

Inserting (9.8), we have

$$\int_V {}^*R^*_{Pij}{}^{\sigma\tau}{:}_\tau \, dV_\sigma \int_V R^{*ij\alpha\beta}_P{:}_\beta \, dV_\alpha = -2\left(\frac{8\pi G}{3}\right)^2 e^{i0k0} \int_V G_{Pi}\, dV_0 \int_V G_{Pk}\, dV_0$$
$$= 0$$

since $e^{i0k0} = 0$. Consequently, *the field equations* (8.9) *satisfy identically the homogeneous quantum condition* (5.22b).

Putting (9.11) into the first quantum condition (5.22a) furnishes

$$32\pi^2 n^2 = \left(\frac{8\pi G}{3}\right)^2 e_{mn}{}^{ij} \int_V G_{Pi}\, dV_j \, e^{mnkl} \int_V G_{Pk}\, dV_l$$
$$= \left(\frac{8\pi G}{3}\right)^2 e_{mn}{}^{ij} e^{mnkl} \int_V G_{Pi}\, dV_j \int_V G_{Pk}\, dV_l$$

Expanding the product of the Levi-Civita tensors

$$9n^2/2G^2 = -2(\eta^{ik}\eta^{jl} - \eta^{il}\eta^{jk}) \int_V G_{Pi}\, dV_j \int_V G_{Pk}\, dV_l$$

or

$$\frac{-9n^2}{4G^2} = \int_V G_P{}^k dV^l \int_V G_{Pk} dV_l - \int_V G_P{}^l dV^k \int_V G_{Pk}\, dV_l$$

Using (9.8), this assumes the form

$$\frac{-9n^2}{4G^2} = \int_V G_P{}^k\, dV^0 \int_V G_{Pk}\, dV_0 - \int_V G_P{}^0\, dV^0 \int_V G_{P0}\, dV_0$$
$$= (\partial p_P{}^k/\partial t)(\partial p_{Pk}/\partial t) - (\partial p_P{}^0/\partial t)(\partial p_{P0}/\partial t)$$
$$= (\partial p_P{}^0/\partial t)^2 - (\partial \mathbf{p}_P/\partial t)\cdot(\partial \mathbf{p}_P/\partial t) - (\partial p_P{}^0/\partial t)^2$$

or

$$(\partial \mathbf{p}_P/\partial t)\cdot(\partial \mathbf{p}_P/\partial t) = 9n^2/4G^2$$

Restoring normal units, we obtain the final, most easily interpreted form of the quantization condition on the sources of the gravitational field:

At those isolated points in space where local energy conservation is violated, the production of energy is not arbitrary; rather, the time rate of appearance of three-momentum is governed by

$$\| \partial \mathbf{p}_P / \partial t \| = 3c^4 n / 2G \tag{9.12}$$

where $n = 0, \pm 1, \pm 2, \ldots$.

10. *Comments and Conclusion*

Our analysis has led to the result that with those points in space that act as sources or sinks of energy, we must associate a quantized force, the fundamental quantum of which is $3c^4/2G = 1.82 \times 10^{49}$ dyn. Energetically this amounts to nonconservation at the rate of 5.44×10^{59} erg $\sec^{-1}$. Unfortunately, our reasoning is mute regarding the specifics of energy creation or annihilation. Some general conclusions may be inferred, however,

According to the Dirac veto, $T_{Pi}{}^{\lambda}(D) \equiv 0$ in spite of the fact that $T_{Pi}{}^{\lambda}{}_{:\lambda}(D) \neq 0$. This must mean that regardless of the form in which energy is created, the energy cannot accumulate at the source point; rather, it must be ejected into the surrounding space as fast as it appears. In order words, *energy cannot appear in the form of a simple stationary rest mass located at the dual source point.*

In addition, experience based on normal general relativity would suggest that if a given volume of space has a time-dependent energy content, the metric surrounding that volume should also be time dependent. This leads us to expect that outgoing gravitational radiation would accompany energy nonconservation events. In turn, this raises the question of the distribution of the newly created energy between gravitational waves and the nongravitational modes of energy existence. The resolution of this question must await the discovery of a specific solution of the field equations (8.9) corresponding to a case of energy nonconservation.

The physical conditions that might trigger energy production are also unknown. The fact that the quantum condition (9.12) involves a three-force does suggest, though, that a prerequisite for energy creation or destruction might be situations giving rise to force fields of the order of c^4/G. A possible example is stellar collapse through the Schwarzschild radius, where tidal forces assume infinite values in a finite proper time. Violation of energy conservation might therefore be expected to play a significant role in black hole formation.

For $n = 1$, (9.12) gives a rate of energy production of roughly 10^5 solar masses per second. It becomes abundantly clear why energy conservation has always been verified, at least locally: an energy creation event near the Earth that proceeded for more than the tiniest fraction of a second would yield a mass so huge as to totally disrupt the solar system as we know it.

The situation in the depths of space is perhaps more promising. It is interesting that in most realistic cosmologies, c^3/G is of the order of the total mass of the universe divided by its age; we have here the possibility of resurrecting a steady-state cosmology without recourse to the cosmic field of Hoyle (1948, 1949).

But where specifically in the cosmos might continuous creation occur? The quasistellar objects come immediately to mind. Unfortunately, even the brightest of the quasars is estimatedto be 13 orders of magnitude less energetic than the 10^{59} erg sec^{-1} required by (9.12). If energy nonconservation *is* the fundamental mechanism underlying quasars, then either that mechanism is switched on only an extremely small fraction of the time, or else by far the bulk of the energy is produced in hard to detect, nonelectromagnetic forms.

All in all, it would seem premature to seriously advance the results of this article as a plausible explanation for any specific astrophysical phenomenon. We do suggest that any experimental evidence adduced in support of energy nonconservation will most probably be the result of astronomical events occurring, we hope and trust, at astronomical distances.

A final caveat is in order. At this stage of our understanding of dual charge, we have no more or less reason to believe in the reality of gravitational dual sources than we do in the existence of magnetic monopoles, and on that subject, with the sole exception of the observed quantization of electric charge, all empirical evidence is negative.

Appendix: Proof of Equation (5.6)

Using the definitions (5.3) and (5.4) of $\boldsymbol{\rho}$, the relation (5.6) may be rewritten

$$M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho}) = 1 + \int_\Sigma M\left[\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}} - \frac{\boldsymbol{\rho}}{\rho}\frac{\partial\rho}{\partial S^{\mu\nu}}\right) + \frac{i\sinh\rho}{\rho}\left(\boldsymbol{\rho}\times\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\right]\cdot\boldsymbol{\tau}\,dS^{\mu\nu} \tag{A1}$$

Noting the identity

$$\partial\boldsymbol{\rho}/\partial S^{\mu\nu} = (\boldsymbol{\rho}/\rho)\,\partial\rho/\partial S^{\mu\nu} + \rho\,\partial(\boldsymbol{\rho}/\rho)/\partial S^{\mu\nu} \tag{A2}$$

(A1) becomes

$$M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho}) = 1 + \int_\Sigma M\left[\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\left(\rho\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}}\right) + \frac{i\sinh\rho}{\rho}\left(\boldsymbol{\rho}\times\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\right]\cdot\boldsymbol{\tau}\,dS^{\mu\nu} \tag{A3}$$

Obviously, the derivative with respect to $S^{\mu\nu}$ of the right-hand side of (A3) is

$$M\left[\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\left(\rho\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}}\right) + \frac{i\sinh\rho}{\rho}\left(\boldsymbol{\rho}\times\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\right]\cdot\boldsymbol{\tau} \tag{A4}$$

Consider now the derivative of the left-hand side of (A3):

$$\partial[M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]/\partial S^{\mu\nu} = (\partial M/\partial S^{\mu\nu})\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho}) + M\,\partial[\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]/\partial S^{\mu\nu} \tag{A5}$$

According to (4.19) and (5.4), we have

$$\partial M/\partial S^{\mu\nu} = \tfrac{1}{4}M\boldsymbol{\tau}\cdot\mathbf{R}_{P\mu\nu} = M\boldsymbol{\tau}\cdot\partial\boldsymbol{\rho}/\partial S^{\mu\nu}$$

Also,

$$\frac{\partial[\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]}{\partial S^{\mu\nu}} = \partial\left(\cosh\rho - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\sinh\rho\right)/\partial S^{\mu\nu}$$

$$= \sinh\rho\,\frac{\partial\rho}{\partial S^{\mu\nu}} - \sinh\rho\,\boldsymbol{\tau}\cdot\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}} - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\cosh\rho\,\frac{\partial\rho}{\partial S^{\mu\nu}}$$

Therefore, (A5) becomes

$$\frac{\partial[M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]}{\partial S^{\mu\nu}} = M\boldsymbol{\tau}\cdot\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\left(\cosh\rho - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\sinh\rho\right)$$

$$+M\left[\sinh\rho\,\frac{\partial\rho}{\partial S^{\mu\nu}} - \sinh\rho\,\boldsymbol{\tau}\cdot\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}} - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\cosh\rho\,\frac{\partial\rho}{\partial S^{\mu\nu}}\right] \tag{A6}$$

Using (3.8) to expand the product of Pauli matrices appearing in the first term on the right-hand side of (A6), we find

$$\frac{\partial[M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]}{\partial S^{\mu\nu}} = M\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\cosh\rho$$

$$-M\left\{\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\cdot\left(\frac{\boldsymbol{\rho}}{\rho}\right) + i\boldsymbol{\tau}\cdot\left[\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\times\frac{\boldsymbol{\rho}}{\rho}\right]\right\}\sinh\rho$$

$$+M\left[\sinh\frac{\partial\rho}{\partial S^{\mu\nu}} - \sinh\rho\,\boldsymbol{\tau}\cdot\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}} - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\cosh\rho\,\frac{\partial\rho}{\partial S^{\mu\nu}}\right]$$

Gathering terms,

$$\frac{\partial[M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]}{\partial S^{\mu\nu}} = M\left\{\left(\boldsymbol{\tau}\cdot\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}} - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\rho}}{\rho}\frac{\partial\rho}{\partial S^{\mu\nu}}\right)\cosh\rho\right.$$

$$\left.+\left[-\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\cdot\frac{\boldsymbol{\rho}}{\rho} - i\boldsymbol{\tau}\cdot\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\times\frac{\boldsymbol{\rho}}{\rho}\right) + \frac{\partial\rho}{\partial S^{\mu\nu}} - \boldsymbol{\tau}\cdot\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}}\right]\sinh\rho\right\} \tag{A7}$$

Observe that

$$(\partial\boldsymbol{\rho}/\partial S^{\mu\nu})\cdot(\boldsymbol{\rho}/\rho) = (2\rho)^{-1}\,\partial(\boldsymbol{\rho}\cdot\boldsymbol{\rho})/\partial S^{\mu\nu} = (2\rho)^{-1}\,\partial(\rho^2)/\partial S^{\mu\nu}$$

$$= (2\rho)^{-1}\,(2\rho\,\partial\rho/\partial S^{\mu\nu}) = \partial\rho/\partial S^{\mu\nu} \tag{A8}$$

Inserting (A8) and (A2) into (A7), we obtain

$$\frac{\partial[M\exp(-\boldsymbol{\tau}\cdot\boldsymbol{\rho})]}{\partial S^{\mu\nu}} = M\left\{\rho\left[\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}}\right]\cdot\boldsymbol{\tau}\cosh\rho + \left[-\frac{\partial\rho}{\partial S^{\mu\nu}}\right.\right.$$

$$\left.\left. - i\boldsymbol{\tau}\cdot\left(\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\times\frac{\boldsymbol{\rho}}{\rho}\right) + \frac{\partial\rho}{\partial S^{\mu\nu}} - \boldsymbol{\tau}\cdot\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}}\right]\sinh\rho\right\}$$

$$= M\left[\left(\cosh\rho - \frac{\sinh\rho}{\rho}\right)\rho\frac{\partial(\boldsymbol{\rho}/\rho)}{\partial S^{\mu\nu}} + \frac{i\sinh\rho}{\rho}\left(\boldsymbol{\rho}\times\frac{\partial\boldsymbol{\rho}}{\partial S^{\mu\nu}}\right)\right]\cdot\boldsymbol{\tau} \tag{A9}$$

which is identical to (A4), thereby establishing the validity of (5.6).

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